Topological properties of Abelian and non-Abelian quantum Hall states classified using patterns of zeros

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It has been shown that different Abelian and non-Abelian fractional quantum Hall states can be characterized by patterns of zeros described by sequences of integers $\{S_a\}$. In this paper, we will show how to use the data $\{S_a\}$ to calculate various topological properties of the corresponding fraction quantum Hall state, such as the number of possible quasiparticle types and their quantum numbers, as well as the actions of the quasiparticle tunneling and modular transformations on the degenerate ground states on torus.

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I. INTRODUCTION

Characterization and classification of wave functions with infinite variables are the key to gain a deeper understanding of quantum phases and quantum phase transitions. For a long time, people have used symmetry properties of the wave functions and the change in those symmetries to understand quantum phases and quantum phase transitions. However, the discovery of fractional quantum Hall (FQH) states[1,2] suggests that the symmetry characterization is not enough to describe different FQH states. Thus FQH states contain a new kind of order-topological order.[3,4] Topological order is so new that the conventional mathematical tools and the language developed for symmetry-breaking orders are not adequate to describe it. Thus, looking for new mathematical framework to describe the topological order becomes an important task in advancing the theory of topological order.[5–8] Intuitively, topological order corresponds to a pattern of long-range quantum entanglement[9] in the ground state, as revealed by the topological entanglement entropy.[10,11] Characterizing topological order is characterizing patterns of long-range entanglement.

In a recent paper,[12] the pattern of zeros is introduced to characterize and classify symmetric polynomials of infinite variables. Applying this result to FQH states, we find that the pattern of zeros characterizes and classifies many Abelian and non-Abelian FQH states. In other words, the pattern of zeros characterizes the topological order in FQH states.[4] This may lead to a deeper understanding of topological order in FQH states.

More concretely, a pattern of zeros that characterizes FQH ground-state wave functions is described by a sequence of integers $\{S_a\}$. Describing pattern of zeros through a sequence of integers $\{S_a\}$ generalizes the pseudopotential construction of ideal Hamiltonian and the associated zero-energy ground state.[13–18] All the topological properties of FQH states should be determined by the data $\{S_a\}$. In this paper, we will show how to use the data $\{S_a\}$ to calculate those topological properties. They include the quantum number of possible quasiparticles and the ground-state degeneracy on torus.[19,20] We also study the actions of quasiparticle tunneling and the modular transformation on the degenerate ground states.[21] The main results of the paper are summarized in Sec. V.

In another attempt to understand the topological order in FQH states, the Jack polynomials are used and various topological properties are calculated.[22–24] Many of those FQH states correspond to nonunitary conformal field theory (CFT) whose stability is questionable and needs to be investigated.[25] The unitary Jack-polynomial states are the Z_n parafermion states which are special cases of the FQH states constructed through the pattern of zeros.

II. FUSION AND PATTERN OF ZEROS

In Ref. 12, we have introduced the pattern of zeros of a wave function by bringing a variables together. In this section, we will review the pattern of zeros within such an approach but from a slightly different angle. Readers who are interested in the new results can go directly to Sec. III.

A. Fusion of a variables

To bring a variables in a symmetric polynomial $\Phi(\{z_i\})$ together, let us view $z_1,\ldots,z_a$ as variables and $z_{a+1},z_{a+2},\ldots$ as fixed parameters. We write $\Phi(z_1,z_2,\ldots)$ as

$$\Phi(\{z_i\}) = \Phi(z_1,\ldots,z_a;z_{a+1},z_{a+2},\ldots),$$

where $\Phi(z_1,\ldots,z_a;z_{a+1},z_{a+2},\ldots)$ is a symmetric polynomial of $z_1,\ldots,z_a$ parametrized by $(z_{a+1},z_{a+2},\ldots)$. Now let us rewrite

$$z_i = \lambda\xi_i + z^{(a)}, \quad z^{(a)} = \frac{z_1 + \cdots + z_a}{a},$$

$$\sum_{i=1}^{a} \xi_i = 0, \quad i = 1,\ldots,a$$

and expand $\Phi(\lambda\xi_1+z^{(a)},\ldots,\lambda\xi_a+z^{(a)};z_{a+1},z_{a+2},\ldots)$ in powers of λ,

$$\Phi(\lambda\xi_1 + z^{(a)}, \ldots, \lambda\xi_a + z^{(a)}; z_{a+1}, z_{a+2}, \ldots) = \sum_{m=0}^{\infty} \lambda^m P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots],$$

where $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$ is a homogeneous symmetric polynomial of $\xi_1, \ldots, \xi_a$ of degree m. $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$ is also a polynomial of $z^{(a)}$ and $z_{a+1}, z_{a+2}, \ldots$. The minimal power of $z^{(a)}$ is zero.

The polynomial $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$ is derived from $\Phi(\{z_i\})$ after fusing a z_i variables into a single $z^{(a)}$ variable. Different choices of $\xi_1, \ldots, \xi_a$ represent different ways to fuse $z_1, \ldots, z_a$ into $z^{(a)}$. As a polynomial of $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$, different ways of fusion can produce linearly independent $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$. Thus $(\xi_1, \ldots, \xi_a)$ can be viewed as a label for those linearly independent polynomials of $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$.

$P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$ is also a symmetric polynomial of $\xi_1, \ldots, \xi_a$. Different $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$ also lead to different symmetric polynomials of $\xi_1, \ldots, \xi_a$. Let $K_{a,m}$ be the number of linearly independent symmetric polynomials of $\xi_1, \ldots, \xi_a$, $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$, produced by all the choices of $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$. Let $F^{(m)}_{\alpha^{(a,m)}}(\xi_1, \ldots, \xi_a)$ be a basis of those symmetric polynomials of $\xi_1, \ldots, \xi_a$, where $\alpha^{(a,m)} = 1, 2, \ldots, K_{a,m}$ label those linearly independent symmetric polynomials of $\xi_1, \ldots, \xi_a$. Thus the different polynomials of $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$ labeled by $(\xi_1, \ldots, \xi_a)$ can be written as

$$P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots] = \sum_{\alpha^{(a,m)}=1}^{K_{a,m}} F^{(m)}_{\alpha^{(a,m)}}(\xi_1, \ldots, \xi_a) P_m[z^{(a)}, \alpha^{(a,m)}; z_{a+1}, z_{a+2}, \ldots].$$

We see that as different polynomials of $z^{(a)}, z_{a+1}, z_{a+2}, \ldots$ labeled by $\xi_1, \ldots, \xi_a$, $P_m[z^{(a)}, (\xi_1, \ldots, \xi_a); z_{a+1}, z_{a+2}, \ldots]$ can also be labeled by $\alpha^{(a,m)}$. $F^{(m)}_{\alpha^{(a,m)}}(\xi_1, \ldots, \xi_a)$ can be viewed as a conversion factor between the two different labels, $\alpha^{(a,m)}$ and $\xi_1, \ldots, \xi_a$.

Let S_a be the minimal total power of $z_1, \ldots, z_a$ in $\Phi(\{z_i\})$. We see that $K_{a,m} = 0$ if $m < S_a$ and

$$K_a \equiv K_{a,S_a} \neq 0.$$

Since the symmetric polynomial $F^{(m)}_{\alpha^{(a,m)}}(\xi_1, \ldots, \xi_a)$ of order $m = 1$ must have a form

$$F^{(m=1)}_{\alpha^{(a,m)}}(\xi_1, \ldots, \xi_a) \propto \sum \xi_i,$$

which vanishes due to the $\Sigma\xi_i = 0$ condition, we find that

$$S_a \neq 1. \tag{1}$$

Here we would like to introduce a unique-fusion condition. If $K_{a,m_a} = 1$ for all a and allowed m_a, then we say that the symmetric polynomial $\Phi(\{z_i\})$ satisfies the unique-fusion condition. In this paper, we will mainly concentrate on symmetric polynomials $\Phi(\{z_i\})$ that satisfy the unique-fusion condition. In this case we can drop the $\alpha_i^{(a,m)}$ labels.

Note that S_1 is the minimum power of z_i in $\Phi(\{z_i\})$. If $S_1 > 0$, we can construct another symmetric polynomial with less total power of z_i,

$$\tilde{\Phi}(\{z_i\}) = \Phi(\{z_i\}) / \prod_i z_i^{S_1}.$$

Since $\Pi_i z_i^{S_1}$ is nonzero except at $z = 0$, $\Phi(\{z_i\})$ and $\Phi(\{z_i\}) / \Pi_i z_i^{S_1}$ have the same pattern of zeros. Since the minimal power of z_i in $\tilde{\Phi}(\{z_i\})$ is zero, without loss of generality, we will assume

$$S_1 = 0. \tag{2}$$

For any symmetric polynomial, we obtain a sequence of integers $S_1, S_2, \ldots$. Different types of symmetric polynomials may lead to different sequences of integers $S_1, S_2, \ldots$. Thus we can use such sequences to characterize different symmetric polynomials. Since S_a describes how fast a symmetric polynomial $\Phi(\{z_i\})$ approaches to zero as we bring a variables together, we will call $S_1, S_2, \ldots$ a pattern of zeros.

For the $\nu = 1/q$ Laughlin state (3), the holomorphic part of the wave function is given by

$$\Phi_{1/q}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^q. \tag{3}$$

We find that S_a is

$$S_a = \frac{qa(a-1)}{2}. \tag{4}$$

For the Laughlin state, $K_{a,m_a} = 0$ or 1. Thus the Laughlin state is a state that satisfies the unique-fusion condition.

B. Derived polynomials

In Sec. II A, 27we have obtained a new polynomial $P(z^{(a)}, \alpha^{(a)}, z_{a+1}, \ldots)$ from $\Phi(\{z_i\})$ by fusing a variables together. We can fuse many clusters of variables together and obtain, in a similar way, a derived polynomial,

$$P(\{z_i^{(a)}, \alpha_i^{(a)}\}),$$

where $z_i^{(a)}$ is a type-a variable obtained by fusing a z_i variables together. Since different ways to fuse z_i's into a $z^{(a)}$ may lead to different derived polynomials, the index $\alpha_i^{(a)}$ is introduced to label those different derived polynomials.

As we let $z_1^{(a)} \to z_1^{(b)}$, we obtain

$$P(z_1^{(a)}, \alpha_1^{(a)}; z_1^{(b)}, \alpha_1^{(b)}; \ldots)\big|_{z_1^{(a)} \to z_1^{(b)} \equiv z^{(a+b)}} \sim (z_1^{(a)} - z_1^{(b)})^{D_{ab}} \tilde{P}(z^{(a+b)}, \alpha^{a+b}; \ldots) + o[(z_1^{(a)} - z_1^{(b)})^{D_{ab}}]. \tag{5}$$

D_{ab} is another set of data that characterize the symmetric polynomial $\Phi(\{z_i\})$. The two sets of data, S_a and D_{ab}, are related by

$$D_{ab} = S_{a+b} - S_a - S_b. \tag{6}$$

This allows us to use S_a to calculate D_{ab}. It also implies that D_{ab} does not depend on $\alpha_1^{(a)}$, $\alpha_1^{(b)}$, and $\alpha^{(a+b)}$.

To derive Eq. (6), let us write $\Phi(\{z_i\})$ as

$$\Phi(z_1,\ldots,z_{a+b};z_{a+b+1},z_{a+b+2},\ldots)$$
$$=\lambda^{S_a}\tilde{\lambda}^{S_b}\sum_{\alpha^{(a)},\alpha^{(b)}}P(z^{(a)},\alpha^{(a)};z^{(b)},\alpha^{(b)};z_{a+b+1},\ldots)$$
$$\times F_{\alpha^{(a)}}(\xi_1,\ldots,\xi_a)F_{\alpha^{(b)}}(\xi_{a+1},\ldots,\xi_{a+b})+\cdots, \quad (7)$$

where

$$z^{(a)}=(z_1+\cdots+z_a)/a, \quad z^{(b)}=(z_{a+1}+\cdots+z_{a+b})/b,$$

$$\lambda\xi_i=z_i-z^{(a)} \quad \text{for } i=1,\ldots,a,$$

$$\tilde{\lambda}\xi_i=z_i-z^{(b)} \quad \text{for } i=a+1,\ldots,a+b.$$

In Eq. (7), $(\cdots)$ represents the higher order terms in λ and $\tilde{\lambda}$. Since in $\Phi(\{z_i\})$ the minimal total power of $z_1,\ldots,z_a$ is S_a and the minimal total power of $z_{a+1},\ldots,z_{a+b}$ is S_b, as a result $F_{\alpha^{(a)}}(\xi_1,\ldots,\xi_a)$ and $F_{\alpha^{(b)}}(\xi_{a+1},\ldots,\xi_{a+b})$ are homogeneous polynomials of degrees S_a and S_b, respectively. Since the minimal total power of $z_1,\ldots,z_{a+b}$ is S_{a+b}, thus the minimal total power of $z_1^{(a)}$ and $z_1^{(b)}$ in $P(z^{(a)},\alpha^{(a)};z^{(b)},\alpha^{(b)};z_{a+b+1},z_{a+b+2},\ldots)$ is $S_{a+b}-S_a-S_a$. This proves Eq. (6).

Equation (6) expresses D_{ab} in terms of S_a. We can also express S_a in terms of D_{ab}. Since $S_1=0$, we find that

$$S_{a+1}=S_a+D_{a,1},$$

which leads to

$$S_a=\sum_{b=1}^{a-1}D_{b,1}. \quad (8)$$

C. *n*-cluster condition

In Ref. 12, we have introduced an n-cluster condition on symmetric polynomials $\Phi(\{z_i\})$ to make polynomials of infinite variables to behave more like polynomials of n variables. Here we will introduce the n-cluster condition in a slightly different way. A symmetric polynomial $\Phi(\{z_i\})$ satisfies the n-cluster condition if and only if (a) the fusion of kn variables is unique, $K_{kn,S_{kn}}=1$, where k=integer, and (b) as a function of a kn-cluster variable $z^{(kn)}$, the derived polynomial $P(z^{(kn)},z_1^{(a)},\ldots)$ is nonzero if $z^{(kn)}\neq z_i^{(a)}$ (i.e., no zeros off the variables $z_i^{(a)}$).

Let

$$m\equiv D_{n,1}, \quad (9)$$

condition (b) allows us to show that

$$D_{kn,a}=kam.$$

From $D_{n,a}=S_{a+n}-S_a-S_n$, we find that $S_{a+n}=S_a+S_n+ma$ which implies that

$$S_{a+kn}=S_a+kS_n+mn\frac{k(k-1)}{2}+kma. \quad (10)$$

Equation (10) is the defining relation that defines the n-cluster condition on the pattern of zeros $\{S_a\}$. We see that $\{S_2,\ldots,S_n\}$ determine the whole sequence $\{S_a\}$.

D. Occupation description of pattern of zeros

There are other ways to encode the data $\{S_a\}$ which can be very useful. In the following, we will introduce occupation description of the pattern of zeros $\{S_a\}$. We first assume that there are many orbitals labeled by $l=0,1,2,\ldots$. Some orbitals are occupied by particles and others are not occupied. The pattern of zeros $\{S_a\}$ corresponds to a particular pattern of occupation. To find the corresponding occupation pattern, we introduce

$$l_a=S_a-S_{a-1}, \quad a=1,2,\ldots \quad (11)$$

and interpret l_a as the label of the orbital that is occupied by the ath particle. Thus the sequence of integers l_a describes a pattern of occupation. l_a is a monotonically increasing sequence,

$$l_{a+2}-l_{a+1}=S_{a+2}+S_a-2S_{a+1}\geq 0,$$

as implied by the condition (see Ref. 12)

$$S_{a+b+c}-S_{a+b}-S_{a+c}-S_{b+c}+S_a+S_b+S_c\geq 0 \quad (12)$$

with $b=c=1$. The n-cluster condition (10) becomes

$$l_{a+n}=l_a+m. \quad (13)$$

So $\{l_1,\ldots,l_n\}$ determine the whole sequence $\{l_a\}$. Two sequences, $\{l_1,\ldots,l_n\}$ and $\{S_2,\ldots,S_n\}$, have a one-to-one correspondence. Thus $\{S_a\}$ and $\{l_a\}$ are faithful representations of each other.

The occupation distribution can also be described by another sequence of integers n_l. Here n_l is the number of l_a's whose value is l. n_l is the number of particles occupying the orbital l. Both $\{l_a\}$ and $\{n_l\}$ are equivalent occupation descriptions of the pattern of zeros $\{S_a\}$.

The occupation distribution $\{n_l\}$ (or equivalently $\{l_a\}$) that describes the pattern of zeros in Φ has a very physical meaning. If we regard n_l as the occupation number on an orbital described by one-body wave function $\phi_l=z^l$, then the occupation distribution $\{n_l\}$ will correspond to a many-boson state described by a symmetric polynomial $\Phi_{\{n_l\}}$. The two many-boson states Φ and $\Phi_{\{n_l\}}$ will have similar density distribution, in particular, in the region far from $z=0$.

E. Ideal Hamiltonian and zero-energy ground state

For an electron system on a sphere with N_ϕ flux quanta, each electron carries an orbital angular momentum $J=N_\phi/2$ if the electrons are in the first Landau level.[13] For a cluster of a electrons, the maximum allowed angular momentum is aJ. However, for the wave function $\Phi(\{z_i\})$ described by the pattern of zeros $\{S_a\}$, the maximum allowed angular momentum for an a-electron cluster in $\Phi(\{z_i\})$ is only $J_a=aJ-S_a$. The pattern of zeros forbids the appearance of angular momenta $aJ-S_a+1,aJ-S_a+2,\ldots,aJ$ for any a-electron clusters in $\Phi(\{z_i\})$.

Such a property of the wave function $\Phi(\{z_i\})$ can help us to construct an ideal Hamiltonian such that Φ is the exact zero-energy ground state.[13–18] The ideal Hamiltonian has the following form:

$$H_{\{S_a\}} = \sum_a \sum_{a\text{-electron clusters}} (P^{(a)}_{S_a} + P_{\overline{\mathcal{H}}_a}), \tag{14}$$

where $P^{(a)}_S$ is a projection operator that projects onto the subspace of a electrons with total angular momenta $aJ-S+1,\ldots,aJ$ and $P_{\overline{\mathcal{H}}_a}$ is a projection operator into the space $\overline{\mathcal{H}}_a$.

What is the space $\overline{\mathcal{H}}_a$? Let us fix $z_{a+1}, z_{a+2}, \ldots$ and view the wave function $\Phi(z_1,\ldots,z_N)$ as a function of $z_1,\ldots,z_a$. Such a wave function describes an a-electron state. If $\Phi(z_1,\ldots,z_N)$ is described by a pattern of zeros $\{S_a\}$, the a-electron state defined above can have a nonzero projection into the space $\mathcal{H}_{a,S_a}$, where $\mathcal{H}_{a,S_a}$ is a space with a total angular momentum $aJ-S_a$. However, different positions of other electrons may lead to different projections of the a-electron states in the space $\mathcal{H}_{a,S_a}$. $\mathcal{H}_a$ is then the subspace of $\mathcal{H}_{a,S_a}$ that is spanned by those states. $\overline{\mathcal{H}}_a$ is the subspace of $\mathcal{H}_{a,S_a}$ formed by vectors that are orthogonal to $\mathcal{H}_a$.

We see that by construction, Φ described by a pattern of zeros $\{S_a\}$ is a zero-energy ground state of the ideal Hamiltonian $H_{\{S_a\}}$. However, the zero-energy states of $H_{\{S_a\}}$ are not unique. In fact, any state $\tilde{\Phi}$ with a pattern of zeros $\{\tilde{S}_a\}$ will be a zero-energy eigenstate of $H_{\{S_a\}}$ if $\tilde{S}_a \geq S_a$. (If $\tilde{S}_a = S_a$, we will further require that $H_a \subseteq \tilde{H}_a$.)

But on sphere, $H_{\{S_a\}}$ may have a unique ground state. If the number of electrons is a multiple of n, $N=nN_c$, and the number of the flux quanta N_ϕ satisfies

$$2J = N_\phi = \frac{2S_n}{n} + m(N_c - 1), \tag{15}$$

then the state Φ can be put on a sphere and corresponds to a unique uniform state with zero total angular momentum. Such a state may be the unique zero-energy ground state of the ideal Hamiltonian $H_{\{S_a\}}$ on the sphere.

If we increase N_ϕ (but fix the number of particles N), then $H_{\{S_a\}}$ will have more zero-energy states. Those zero-energy states can be viewed as formed by a few quasiparticle excitations. Those quasiparticle excitations may carry fraction charges and fractional statistics. In Sec. III, we will study those zero-energy quasiparticles through the pattern of zeros.

III. QUASIPARTICLES

A. Quasiparticles and patterns of zeros

If we let $z_1,\ldots,z_a$ approach 0 in a ground-state wave function $\Phi(\{z_i\})$, we obtain

$$\Phi(\{z_i\})|_{\lambda\to 0} = \lambda^{S_a} P(\xi_1,\ldots,\xi_a; z_{a+1},\ldots) + \cdots,$$

where $z_i = \lambda\xi_i$, $i=1,\ldots,a$. Due to the translation invariance of the polynomial Φ, the pattern of zeros $\{S_a\}$ indicates that the $z=0$ point is the same as any other point, since we will get the same pattern of zeros if we let $z_1,\ldots,z_a$ to approach to any other point. Thus the pattern of zeros $\{S_a\}$ describes a state with no quasiparticle at $z=0$.

If a new symmetric polynomial $\Phi_\gamma(\{z_i\})$ has a quasiparticle at $z=0$, $\Phi_\gamma(\{z_i\})$ will have a different pattern of zeros $\{S_{\gamma;a}\}$ as $z_1,\ldots,z_a$ approach 0,

$$\Phi_\gamma(\{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(\xi_1,\ldots,\xi_a; z_{a+1},\ldots) + \cdots. \tag{16}$$

In other words, the minimal total power of $z_1,\ldots,z_a$ in $\Phi_\gamma(\{z_i\})$ is $S_{\gamma;a}$. Thus we can use $\{S_{\gamma;a}\}$ to characterize different quasiparticles. Here the index γ labels different types of quasiparticles. We would like to stress that the quasiparticle wave function $\Phi_\gamma(\{z_i\})$ is still described by the pattern of zeros $\{S_a\}$ if we let $z_1,\ldots,z_a$ approach to any other point away from $z=0$.

Just as $\{S_a\}$, $\{S_{\gamma;a}\}$ should also satisfy certain conditions. Here, we will consider quasiparticle states $\Phi_\gamma(\{z_i\})$ that have a zero energy for the ideal Hamiltonian $H_{\{S_a\}}$. We note that when we consider topological properties of quasiparticles, we regard that quasiparticles differ by electrons to be equivalent. In every equivalent class of quasiparticles, there are always members that correspond to the zero-energy excitations for the ideal Hamiltonian. (In other words, a quasiparticle, when combined with enough holes, will become a zero energy excitation.) So by studying zero-energy quasiparticles, we can study topological properties of all quasiparticles.

The zero-energy condition on quasiparticles requires that $S_{\gamma;a}$ should satisfy

$$S_{\gamma;a} \geq S_a. \tag{17}$$

Although both the ground state Φ and the quasiparticle states Φ_γ have a zero energy, the ground state has the minimal power of z_i while the quasiparticle states have higher total powers of z_i.

In the following, we will discuss other conditions on $S_{\gamma;a}$. Let $P_\gamma(\{z_i^{(a)},\alpha_i^{(a)}\})$ be the derived polynomial obtained from $\Phi_\gamma(\{z_i\})$. First we let $z^{(a)}\to 0$ in P_γ,

$$P_\gamma(z^{(a)},\alpha^{(a)};z^{(b)},\alpha^{(b)};\ldots) \sim (z^{(a)})^{D_{\gamma;a}} P'_\gamma(\alpha^{(a)};z^{(b)},\alpha^{(b)};\ldots).$$

Then we let $z^{(b)}\to 0$ and let $D_{\gamma;a,b}$ be the order of zeros in $P'_\gamma(\alpha^{(a)};z^{(b)},\alpha^{(b)};\ldots)$ as $z^{(b)}\to 0$,

$$P'_\gamma(\alpha^{(a)};z^{(b)},\alpha^{(b)};\ldots) \sim (z^{(b)})^{D_{\gamma;a,b}} P''_\gamma(\alpha^{(a)};\alpha^{(b)};\ldots).$$

We have

$$S_{\gamma;a+b} = S_{\gamma;a} + D_{\gamma;a,b} + S_b, \tag{18}$$

which leads to a condition on $S_{\gamma;a}$,

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0.$$

To get more conditions on $S_{\gamma;a}$, let us consider fusing three variables $z^{(a)}$, $z^{(b)}$, and $z^{(c)}$ to 0. We may first fuse $z^{(a)}$ to 0 to obtain $P'_\gamma(\alpha^{(a)};z^{(b)},\alpha^{(b)};z^{(c)},\alpha^{(c)};\ldots)$. Then we fuse $z^{(b)}$ to 0 to obtain $P''_\gamma(\alpha^{(a)};\alpha^{(b)};z^{(c)},\alpha^{(c)};\ldots)$. Last we fuse $z^{(c)}$ to 0. In this case, we obtain a $D_{\gamma;a+b,c}$th order zero as $z^{(c)}\to 0$ in $P''_\gamma(\alpha^{(a)};\alpha^{(b)};z^{(c)},\alpha^{(c)};\ldots)$. We also know that P'_γ has a $D_{b,c}$th order zero as $z^{(c)}\to z^{(b)}$ and a $D_{\gamma;a,c}$th order zero as $z^{(c)}\to 0$ (see Fig. 1). If $z^{(b)}$ is very close to 0, we find that $D_{\gamma;a+b,c}$ is the sum of $D_{\gamma;a,c}$, $D_{b,c}$, and the zeros close to $z=0$ but not at $z=0$ and at $z^{(b)}$. This way, we find

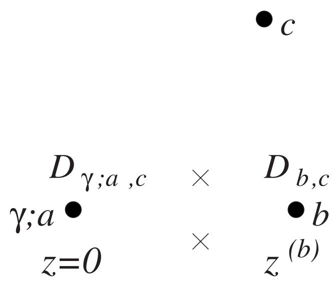

FIG. 1. As a function of $z^{(c)}$, $P'_\gamma(\alpha^{(a)};z^{(b)},\alpha^{(b)};z^{(c)},\alpha^{(c)};\ldots)$ has a $D_{\gamma;a,c}$th order zero as $z^{(c)}\to 0$ and a $D_{b,c}$th order zero as $z^{(c)}\to z^{(b)}$. The crosses mark the positions of other zeros that are not at $z=0$ and not at $z^{(b)}$. The total number of zeros seen by the $z^{(c)}$ in the neighborhood of $z=0$ and $z^{(b)}$ is $D_{\gamma;a+b,c}$, which satisfies Eq. (19).

$$D_{\gamma;a+b,c} \geq D_{\gamma;a,c} + D_{b,c}, \tag{19}$$

which implies that

$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0. \tag{20}$$

Equation (20) generalizes condition (12) on the pattern of zeros of the ground state.

The n-cluster condition implies that, as a function of $z^{(n)}$, $P_\gamma(z^{(n)},z_i^{(a)},\ldots)$ is nonzero if $z^{(n)}\neq z_i^{(a)}$ and $z^{(n)}\neq 0$. Therefore, inequality (19) is saturated when $c=n$,

$$D_{\gamma;a+b,n} = D_{\gamma;a,n} + D_{b,n},$$

which implies that (setting $a=0$)

$$S_{\gamma;b+n} = S_{\gamma;b} + D_{\gamma;0,n} + S_{b+n} - S_b = S_{\gamma;b} + S_n + D_{\gamma;0,n} + mb$$

or

$$S_{\gamma;a+kn} = S_{\gamma;a} + k(S_n + ma + D_{\gamma;0,n}) + mn\frac{k(k-1)}{2}.$$

Since $S_\gamma=0$, we have $S_{\gamma;n}=S_n+D_{\gamma;0,n}$. Thus

$$S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + mn\frac{k(k-1)}{2} \tag{21}$$

for $a\geq 0$. This is the n-cluster condition on $S_{\gamma;a}$.

B. Solutions of patterns of zeros for quasiparticles

The ground state of the ideal Hamiltonian (14) is a state described by a pattern of zero $\{S_a\}$. The zero-energy quasiparticles above the ground state Φ can be characterized by a pattern of zeros $\{S_{\gamma;a}\}$. If Φ is a state with the n-cluster form, then all the $S_{\gamma;a}$'s are determined from $S_{\gamma;1},\ldots,S_{\gamma;n}$ [see Eq. (21)]. Thus zero-energy quasiparticles are labeled by n integers: $S_{\gamma;1},\ldots,S_{\gamma;n}$. From the discussion in Sec. III A, we find that those integers should also satisfy

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0,$$

$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0. \tag{22}$$

Numerical experiments suggest that the solutions of Eq. (22) always satisfy Eq. (17). Thus condition (17) is not included.

For a given pattern of zeros $\{S_a\}$ describing a symmetric polynomial Φ of n-cluster form, Eq. (22) has many solutions that satisfy the n-cluster condition (21). Each of those solutions corresponds to a kind of quasiparticle. Those solutions (and the quasiparticles) can be grouped into equivalent classes.

To describe such equivalent classes, we need to use the occupation description of the pattern of zeros $\{S_{\gamma;a}\}$. We introduced

$$l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}, \quad a=1,2,\ldots$$

and interpreted $l_{\gamma;a}$ as the label of the orbital that is occupied by the ath particle. Thus the sequence of integers $l_{\gamma;a}$ describes a pattern of occupation. We note that $l_{\gamma;a+1}=D_{\gamma;a,1}$. Equation (19) implies that $l_{\gamma;a+1}\geq l_{\gamma;a}$.

The occupation distribution can also be described by another sequence of integers $n_{\gamma;l}$. Here $n_{\gamma;l}$ is the number of $l_{\gamma;a}$'s whose value is l. $n_{\gamma;l}$ is the number of particles occupying the orbital l. Both $\{l_{\gamma;a}\}$ and $\{n_{\gamma;l}\}$ are equivalent occupation descriptions of the pattern of zeros $\{S_{\gamma;a}\}$.

The n-cluster condition on $\{S_{\gamma;a}\}$ implies that

$$l_{\gamma;a+n} = l_{\gamma;a} + m \tag{23}$$

for $a\geq 1$. This implies that occupation numbers $\{n_{\gamma;l}\}$ satisfy

$$n_{\gamma;l+m} = n_{\gamma;l} \tag{24}$$

for a large enough l. Thus the occupation numbers $n_{\gamma;l}$ become a periodic function of l with a period m in large l limit. In that limit, each m orbitals contain n particles.

Now consider two quasiparticles γ_1 and γ_2 described by two sequences of occupation numbers $\{n_l^{\gamma_1}\}$ and $\{n_l^{\gamma_2}\}$. If $n_{\gamma_1;l}=n_{\gamma_2;l}$ for large enough l, then the density distributions of two many-boson states Φ_{γ_1} and Φ_{γ_2} only differ by an integral number of electrons near $z=0$. Thus the two quasiparticles γ_1 and γ_2 differ only by an integral number of electrons. This motivates us to say that the two quasiparticles are equivalent.

Numerical experiments suggest that each equivalent class is represented by a simple occupation distribution $n_{\gamma;l}$ that is periodic for all $l>0$,

$$n_{\gamma;l+n} = n_{\gamma;l}, \quad l\geq 0. \tag{25}$$

We call such a distribution the canonical distribution for the corresponding equivalent class. For a canonical occupation distribution, there are n particles in every unit cell ($l=0,\ldots,m-1$), ($l=m,\ldots,2m-1$),….

We can obtain all the equivalent classes of the quasiparticles by finding all the canonical distributions that satisfy Eq. (22). The equivalent classes of the quasiparticles correspond to fractionalized excitations. This way, we find all the quasiparticle types for a FQH state described by a pattern of zeros $\{S_a\}$.

IV. TOPOLOGICAL PROPERTIES FROM PATTERN OF ZEROS

A FQH state characterized by a pattern of zeros $\{S_a\}$ can have many topological properties, such as quasiparticle quantum numbers,[2,26] ground-state degeneracy on compact space,[3,20,27] and edge excitations.[28] In this section, we are going to calculate some of those topological properties from the data $\{S_a\}$.

A. Charge of quasiparticles

We have seen that a quasiparticle excitation labeled by γ is characterized by a sequence of integers: $\{S_{\gamma;a}\}$. We would like to calculate the quantum numbers of the quasiparticle from $\{S_{\gamma;a}\}$.

To calculate the quasiparticle charge, we compare the occupations n_l that describe the pattern of zeros $\{S_a\}$ of the ground state Φ and the occupations $n_{\gamma;l}$ that describe the pattern of zeros $\{S_{\gamma;a}\}$ of a quasiparticle state Φ_γ. We divide $l=0,1,2,\ldots$ into unit cells each containing m orbitals: $l=0,\ldots,m-1;m,\ldots,2m-1;\ldots$. n_l and $n_{\gamma;l}$ contain the same number of particles in the kth unit cell if k is large enough. Since n_l, describing a null quasiparticle, is a distribution that corresponds to zero quasiparticle charge, we might think that the quasiparticle charge corresponding to the distribution $n_{\gamma;l}$ is given by

$$\sum_{l=0}^{km}(n_l-n_{\gamma;l})$$

in large k limit. However, this result is incorrect. Although n_l and $n_{\gamma;l}$ contain the same n particles in the kth unit cell (for a large k), the "centers of mass" of the two distributions in the kth cell are different. The shift of the centers of mass is given by

$$\frac{1}{m}\sum_{l=km-m}^{km-1}(n_l-n_{\gamma;l})l.$$

Shifting the center of mass by m is equivalent to adding or removing n particles. Thus the total quasiparticle charge is given by

$$Q_\gamma=\sum_{l=0}^{km}(n_l-n_{\gamma;l})-\frac{1}{m}\sum_{l=km-m}^{km-1}(n_l-n_{\gamma;l})l \tag{26}$$

for a large enough k. Note that in the above definition, a charge +1 corresponds to an absence of an electron. For a canonical occupation distribution satisfying Eq. (25), the first term $\Sigma_{l=0}^{km}(n_l-n_{\gamma;l})$ vanishes.

Since the two descriptions of occupation distributions, $\{l_{\gamma;a}\}$ and $\{n_{\gamma;l}\}$, have a one-to-one correspondence, we can also express Q_γ in terms of $\{l_{\gamma;1},\ldots,l_{\gamma;n}\}$ (note that, according to Eq. (23), $\{l_{\gamma;1},\ldots,l_{\gamma;n}\}$ determines the whole sequence $\{l_{\gamma;a}\}$),

$$Q_\gamma=\frac{1}{m}\sum_{a=1}^{n}(l_{\gamma;a}-l_a). \tag{27}$$

There is another way to calculate the charge of a quasiparticle. We can put the quasiparticle state Φ_γ on a sphere with N_ϕ flux quanta. If we move the quasiparticle around a loop that spans a solid angle Ω, the quasiparticle state Φ_γ will generate a Berry phase,

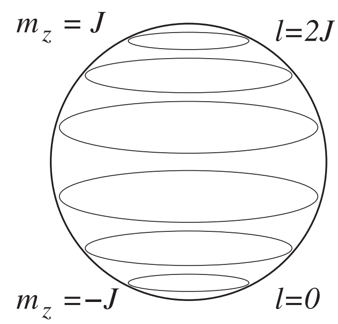

FIG. 2. The $2J+1$ orbital on a sphere forms an angular-momentum J representation of the $O(3)$ rotation.

$$\theta_B(\Omega)=\frac{N_\phi\Omega}{2}Q_\gamma+O(N_\phi^0). \tag{28}$$

The part in θ_B that is proportional to N_ϕ allows us to determine the charge Q_γ.

From the Berry phase $\theta_B(\Omega)$ (as a function of the solid angle Ω), we can find out the angular momentum J_q of the quasiparticle,

$$J_q=\theta_B(\Omega)/\Omega$$

(note that $\theta_B\propto\Omega$). The occupation distribution n_l for the ground state Φ describes a trivial quasiparticle with zero charge and zero angular momentum. The occupation distribution $n_{\gamma;l}$ for the quasiparticle state Φ_γ describes a nontrivial quasiparticle with a nonzero angular momentum J_q. Since the orbital ϕ_l, when put on a sphere with N_ϕ flux quanta, is identified with an angular-momentum eigenstate $(J,J_z)=(\frac{N_\phi}{2},l-J)$, the quasiparticle state Φ_γ is an eigenstate of total J_z^{tot} (see Fig. 2). Since Φ_γ is the state with maximal J_z^{tot}, the total angular momentum of the quasiparticle is $J^{\text{tot}}=J_z^{\text{tot}}\equiv J_q$.

Since the occupation distribution n_l describes a state with $J_z^{\text{tot}}=0$, the angular momentum of the quasiparticle is

$$J_q=\sum_{l=0}(n_{\gamma;l}-n_l)(l-J)$$

or more generally the Berry phase of the quasiparticle is

$$\theta_B^\gamma=\Omega\sum_{l=0}(n_{\gamma;l}-n_l)(l-J), \tag{29}$$

where the upper bound of the summation $\Sigma_{l=0}$ is roughly at $l=N_\phi=2J$. From the part of θ_B^γ that is linear in $N_\phi=2J$, we can recover Eq. (26) for the charge of the quasiparticle.

B. Orbital spin of quasiparticles

We have seen that the Berry phase of the quasiparticle contains a term linear in $N_\phi=2J$ that is related to the quasiparticle charge. The Berry phase also contains a constant term which by definition determines the orbital spin S_γ^{osp} of the quasiparticle.[4,29,30] More precisely, in the large $N_\phi=2J$ limit, we have

$$\frac{\theta_B^\gamma}{\Omega} = JQ_\gamma + S_\gamma^{\mathrm{osp}} + O(J^{-1}). \tag{30}$$

Therefore, to calculate S_γ^{osp}, we need to evaluate Eq. (29) carefully. Equation (29) is not a well defined expression since the upper bound of the summation $\Sigma_{l=0}$ is not given precisely (on purpose). So to evaluate Eq. (29), we first regulate Eq. (29) as

$$\theta_B^\gamma = \Omega \sum_{l=0} (n_{\gamma;l} - n_l)(l-J)e^{-\alpha^2 l^2} \tag{31}$$

and then take the small α and large J limit with $\alpha \sim 1/J$. The key difference between Eqs. (29) and (31) is that Eq. (31) has a soft (or smooth) cutoff near the upper bound of the summation $\Sigma_{l=0}$. After evaluating the regulated summation (31) in Appendix, Sec. 1, we find that the orbital spin of the quasiparticle is given by

$$S_\gamma^{\mathrm{osp}} = \sum_{l=0}^{m-1} (n_{\gamma;l} - n_l)\left(\frac{l}{2} - \frac{l^2}{2m}\right). \tag{32}$$

Equation (32) is valid only for canonical occupation distributions that satisfy Eq. (25). In terms of $l_{\gamma;a}$, we can rewrite Eq. (32) as

$$S_\gamma^{\mathrm{osp}} = \sum_{a=1}^{n} \frac{ml_{\gamma;a} - l_{\gamma;a}^2 - ml_a + l_a^2}{2m}. \tag{33}$$

However, Eq. (33) is more general. It applies to generic quasiparticles even those whose occupation distributions do not satisfy Eq. (25).

We would like to stress that Eq. (33) is obtained through an untested method. Its validity is confirmed only for Abelian quasiparticles (see Appendix, Sec. 2). Independent confirmation is needed for more general cases.

C. Ground-state degeneracy on torus

A FQH state has a topological ground-state degeneracy on torus, which is robust against any local perturbations.[3,20] Such a topological ground-state degeneracy is part of the defining properties of topological orders.

According to topological quantum-field theory,[31–33] the topological ground-state degeneracy is equal to the number of quasiparticle types. To be precise, two quasiparticles are regarded equivalent if they differ by a multiple of electrons. Thus, the topological ground-state degeneracy is equal to the number of equivalent classes of quasiparticles introduced in Sec. III B.

To understand such a result, let us consider a quasiparticle state Φ_γ on a sphere (which is a zero-energy state of the ideal Hamiltonian). We stretch the sphere into a thin long tube. The state Φ_γ remains to be a zero-energy state in such a limit. According to Refs. 34–36, the FQH state Φ_γ in such a limit becomes a charge-density-wave (CDW) state characterized by a particle occupation distribution among the orbitals on the thin tube. We expect such an occupation distribution is given by $n_{\gamma;l}$ in the large l limit. We see that $n_{\gamma;l}$ in the large l limit is a CDW state in the thin cylinder limit that is compatible with the zero-energy requirement. This suggests that the quasiparticle types (determined by $n_{\gamma;l}$ in the large l limit) and the zero-energy CDW states on thin cylinder (given by $n_{\gamma;l}$ in the large l limit) are closely related.

The above result allows us to construct zero-energy ground states on a torus. Let us consider a torus with $N_\phi = N_c m$ flux quanta. There are N_ϕ orbitals on such a torus. Those orbitals are labeled by $l=0,1,2,\ldots,N_\phi-1$. Now let us consider a $N=N_c n$-electron FQH state on such a torus. The FQH state is described by a pattern of zeros $\{S_a\}$ of n-cluster form and has a filling fraction $\nu=n/m$. What are the degenerate ground states of such a FQH state on the torus?

According to Refs. 34–36, the zero-energy ground-state wave functions of a FQH state in the thin cylinder limit can be described by certain occupation distribution patterns (or certain CDW states). The above discussion on sphere suggests that the canonical distributions $n_{\gamma;l}$ for the quasiparticles are just those occupation distributions. (Note that the canonical distributions fill each m orbitals with n electrons and give rise to very uniform distributions.) Thus each canonical distribution $n_{\gamma;l}$ gives rise to a N-electron ground-state wave function on the thin torus which corresponds to a degenerate ground state. We see that the canonical occupation distributions $n_{\gamma;l}$ characterize both the degenerate ground states on torus and different types of quasiparticles. This explains why the ground-state degeneracy on torus is equal to quasiparticle types. We also see that we can use m integers $n_{\gamma;l}$, $l=0,1,\ldots,m-1$, to label different degenerate ground states.

The condition on the CDW distributions, $n_{\gamma;l}$, which correspond to the zero-energy ground states on thin torus, can be stated in the translation invariant way. Using $l_{\gamma;a}$'s, we can rewrite the second expression of Eq. (22) as

$$\sum_{k=1}^{c} (l_{\gamma;a+b+k} - l_{\gamma;a+k}) \geq S_{b+c} - S_b - S_c = D_{b,c}. \tag{34}$$

Any $\{l_{\gamma;a}\}$ (or the corresponding $\{n_{\gamma;l}\}$) that satisfy the above conditions are CDW distributions that correspond to zero-energy ground states on torus. Thus Eq. (34) allows us to calculate the ground-state degeneracy on torus and the number of quasiparticle types for a FQH state described by a pattern of zeros $\{S_a\}$.

To understand the meaning of Eq. (34), let us consider a special case of $c=1$ in Eq. (34). In this case, Eq. (34) requires that a zero-energy CDW distribution must satisfy the following condition: any groups of b electrons must spread over $D_{b,1}+1$ orbitals or more. Condition (34) generalizes those more special conditions introduced in Ref. 34–36.

After knowing the one-to-one correspondence between the quasiparticle types and the degenerate ground states, we like to ask which quasiparticle type corresponds to which ground states? Since both quasiparticle types and the degenerate ground states are labeled by the canonical occupation distribution, one may expect that a quasiparticle labeled by $n_{\gamma;l}$ will correspond to the ground state labeled by $n_{\gamma;l}$. However, this does not have to the case. In general, a quasiparticle labeled by $n_{\gamma;l}$ may correspond to the ground state labeled by $n_{\gamma;l+l_s}$. Later, we see that the shift l_s is indeed nonzero. So we will denote the ground-state wave function that corresponds to a quasiparticle γ as $\Phi_{\{n_{\gamma;l+l_s}\}}(\{X_i\})$ or more briefly as $\Phi_\gamma(\{X_i\})$.

D. Quantum numbers of ground states

The Hamiltonian of FQH state on torus has certain symmetries. The degenerate ground state on torus will form a representation of those symmetries. In this section, we will discuss some of those representations.

1. Hamiltonian on torus

First, let us specify the Hamiltonian of the FQH system more carefully. The kinetic energy of the FQH system is determined by the following one-electron Hamiltonian on a torus $(X^1,X^2)\sim(X^1+1,X^2)\sim(X^1,X^2+1)$ with a general mass matrix,

$$H_X=-\frac{1}{2}\sum_{i,j=1,2}\left(\frac{\partial}{\partial X^i}-iA_i\right)g_{ij}\left(\frac{\partial}{\partial X^j}-iA_j\right), \qquad (35)$$

where g is the inverse-mass matrix,

$$g(\tau)=\begin{pmatrix}\tau_y+\dfrac{\tau_x^2}{\tau_y} & -\dfrac{\tau_x}{\tau_y}\\ -\dfrac{\tau_x}{\tau_y} & \dfrac{1}{\tau_y}\end{pmatrix} \qquad (36)$$

and

$$(A_1,A_2)=(-2\pi N_\phi X^2,0) \qquad (37)$$

gives rise to a uniform magnetic field with N_ϕ flux quanta going through the torus. The state in the first Landau level has a form

$$\phi(X^1,X^2)=f(X^1+\tau X^2)e^{i\pi N_\phi\tau(X^2)^2}, \qquad (38)$$

where $\tau=\tau_x+i\tau_y$ and $f(z)$ is a holomorphic function that satisfies the following periodic boundary condition:

$$f(z+a+b\tau)=f(z)e^{-i\tau\pi b^2N_\phi-i2\pi bN_\phi z}, \quad a,b=\text{integers}. \qquad (39)$$

The above holomorphic functions can be expanded by the following N_ϕ basis wave functions:

$$f^{(l)}(z|\tau)=\sum_k e^{i(\pi\tau/N_\phi)(N_\phi k+l)^2+i2\pi(N_\phi k+l)z}, \qquad (40)$$

where $l=0,\ldots,N_\phi-1$. The corresponding wave functions

$$\phi^{(l)}(X^1,X^2|\tau)=f^{(l)}(X^1+\tau X^2|\tau)e^{i\pi N_\phi\tau(X^2)^2} \qquad (41)$$

are orbitals on the torus.

The thin cylinder limit[34–37] is realized by taking $\tau_y\to\infty$. In such a limit

$$\begin{aligned}\phi^{(l)}(X^1,X^2|\tau)&=f^{(l)}(X^1+\tau X^2|\tau)e^{i\pi N_\phi\tau(X^2)^2}\\ &=\sum_k e^{i\pi\tau N_\phi[k+(l/N_\phi)+X^2]^2+i2\pi(N_\phi k+l)X^1}\\ &\approx e^{i\pi\tau N_\phi\{X^2-[1-(l/N_\phi)]^2\}+i2\pi(l-N_\phi)X^1}.\end{aligned}$$

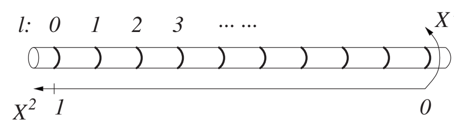

FIG. 3. The circular orbital wave function $\phi^{(l)}(X^1,X^2)$ in the thin cylinder limit. A thick line marks the positions where $\phi^{(l)}(X^1,X^2)$ is peaked.

We see that in the thin cylinder limit, the orbital wave function $\phi^{(l)}(X^1,X^2|\tau)$ is nonzero only near $X^2=1-\frac{l}{N_\phi}$ (see Fig. 3).

2. Translation symmetry

First let us consider the symmetry of Hamiltonian (35). Since the magnetic field is uniform, we expect the translation symmetry in both X^1 and X^2 directions. Hamiltonian (35) does not depend on X^1, thus

$$T^\dagger_{(d_1,0)}HT_{(d_1,0)}=H, \quad T_{(d_1,0)}=e^{d_1\partial_{X^1}}.$$

But Hamiltonian (35) depends on X^2 and there seems no translation symmetry in X^2 direction. However, we do have a translation symmetry in X^2 direction once we include the gauge transformation. Hamiltonian (35) is invariant under $X^2\to X^2+d_2$ transformation *followed* by a $e^{i2\pi N_\phi d_2X^1+i\phi}$ $U(1)$-gauge transformation,

$$T^\dagger_{(0,d_2)}HT_{(0,d_2)}=H, \quad T_{(0,d_2)}=e^{i2\pi N_\phi d_2X^1+i\phi}e^{d_2\partial_{X^2}}.$$

In general,

$$T^\dagger_{\boldsymbol{d}}HT_{\boldsymbol{d}}=H, \quad T_{\boldsymbol{d}}=e^{i2\pi N_\phi d_2X^1+i\pi N_\phi d_1d_2}e^{\boldsymbol{d}\cdot\partial}, \qquad (42)$$

where we have chosen the constant phase in $T_{\boldsymbol{d}}$,

$$\phi=\pi N_\phi d_xd_y, \qquad (43)$$

to simplify the later calculations. The operator $T_{\boldsymbol{d}}$ is called magnetic translation operator. So Hamiltonian (35) does have translation symmetry in any directions. But the (magnetic) translations in different directions do not commute,

$$T_{\boldsymbol{d}}T_{\boldsymbol{d}'}=e^{i2\pi N_\phi(d_1d_2'-d_2d_1')}T_{\boldsymbol{d}'}T_{\boldsymbol{d}}, \qquad (44)$$

and momenta in X^1 and X^2 directions cannot be well defined at the same time.

However, when N_ϕ is an integer, $T_{(1,0)}$ and $T_{(0,1)}$ do commute and the wave function $\phi^{(l)}$ in the first Landau levels satisfies

$$T_{(1,0)}\phi^{(l)}=T_{(0,1)}\phi^{(l)}=\phi^{(l)}.$$

Therefore $\phi^{(l)}$ is a wave function that lives on the torus $(X^1,X^2)\sim(X^1+1,X^2)\sim(X^1,X^2+1)$.

On a torus, the allowed translations are discrete since those translation must commute with $T_{(1,0)}$ and $T_{(0,1)}$. The smallest translations in X^1 and X^2 directions are given by

$$T_1=T_{(1/N_\phi,0)}, \quad T_2=T_{(0,1/N_\phi)},$$

which satisfy

$$T_1T_2 = e^{i2\pi/N_\phi}T_2T_1. \tag{45}$$

We also find that (see Appendix, Sec. 3)

$$T_1\phi^{(l)} = e^{i2\pi l/N_\phi}\phi^{(l)}, \quad T_2\phi^{(l)} = \phi^{(l+1)}.$$

The above T_1 and T_2 act on single-body wave functions. To obtain T_1 and T_2 that act on the many-body ground states, let us consider the many-body ground-state wave functions $\Phi_{\{n_{\gamma;l+l_s}\}}(\{X_i\})=\Phi_\gamma(\{X_i\})$ of the FQH state on torus. Those ground states are labeled by the canonical occupation distributions [see Eq. (25)]. In the thin cylinder limit $\tau_y\to\infty$, the many-body ground-state wave functions $\Phi_{\{n_{\gamma;l+l_s}\}}$ become the CDW functions described by the occupation distributions $\{n_{\gamma;l+l_s}\}$ where there are $n_{\gamma;l+l_s}$ electrons occupying the orbital $\phi^{(l)}$. This allows us to obtain how $\Phi_{\{n_{\gamma;l+l_s}\}}$'s transform under translation,

$$\begin{aligned}T_1\Phi_{\{n_{\gamma;l+l_s}\}} &= \exp^{i2\pi\Sigma_{l=0}^{N_\phi-1}n_{\gamma;l+l_s}l/N_\phi}\Phi_{\{n_{\gamma;l+l_s}\}}\\ &= \exp^{i2\pi\Sigma_{l=0}^{N_\phi-1}n_{\gamma;l}(l-l_s)/N_\phi}\Phi_{\{n_{\gamma;l+l_s}\}}\\ &= \exp^{i2\pi\Sigma_{l=0}^{m-1}n_{\gamma;l}\left(\frac{l}{m}+\frac{N_\phi-2l_s-m}{2m}\right)}\Phi_{\{n_{\gamma;l+l_s}\}}.\end{aligned}$$

If we choose

$$l_s = \frac{N_\phi - m + l_{\max}}{2},$$

which is always an integer since $N_\phi=mN_c$, m=even, and

$$l_{\max} = S_n - S_{n-1} = \text{even}, \tag{46}$$

we find that

$$\begin{aligned}T_1\Phi_\gamma &= \exp^{i2\pi\Sigma_{l=0}^{m-1}(n_{\gamma;l}-n_l)l/m}\Phi_\gamma, = \exp^{i2\pi Q_\gamma}\Phi_\gamma,\\ T_2\Phi_\gamma &= \Phi_{\{n_{\gamma;(l-1)\%m+l_s}\}} = \Phi_{\gamma'},\end{aligned} \tag{47}$$

where we have used $\Sigma_{l=0}^{m-1}\ln_l=nl_{\max}/2$ [see Eq. (A4)]. Here γ' is the quasiparticle described by the canonical occupation distribution $n_{\gamma';l}=n_{\gamma;(l-1)\%m}$. We see that the eigenvalue of T_1 is related to the charge of the corresponding quasiparticle $Q_\gamma=\Sigma_{l=0}^{m-1}(n_{\gamma;l}-n_l)l/m$. The action of T_2 just shifts the occupation distribution by one step. The new distribution describes a new quasiparticle γ'. Those T_1 and T_2 on the many-body wave functions satisfy the following algebra:

$$T_1T_2 = e^{i2\pi n/m}T_2T_1, \quad T_1^m = T_2^m = 1, \tag{48}$$

the degenerate ground states on torus for a representation of the above algebra.

3. Modular transformations

The degenerate ground states on torus form a projective representation of modular transformation,[21]

$$M = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,Z),$$

where $a,b,c,d\in Z$ and $ad-bc=1$ (see Appendix, Sec. 4). The modular representation may contain information that completely characterizes the topological order in the corresponding FQH state. Thus understanding the modular representations may be the key to fully understand the topological order in FQH states.

The modular transformations are generated by

$$M_T = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}. \tag{49}$$

For every $M\in SL(2,Z)$, we have an invertible transformation $U(M)$ acting on the degenerate ground states on torus. $U(M)$'s satisfy

$$U(M)U(M') \sim U(MM'),$$

where $\sim$ mean equal up to a total phase factor. The two generators M_T and M_S are represented as

$$T = U(M_T), \quad S = U(M_S). \tag{50}$$

$M=-1$ is represented as

$$C = U(-1),$$

which is the quasiparticle conjugation operator [see Eq. (A17)].

T, S, T_1, and T_2 have the following algebraic relation (see Appendix, Sec. 5):

$$\begin{aligned}TT_1 &= T_1T, \quad TT_2 = e^{i\pi(n/m)}T_2T_1T,\\ ST_1 &= T_2S, \quad ST_2 = CT_1C^{-1}S.\end{aligned} \tag{51}$$

Since $M_S^2=-1$ and $(M_SM_T)^3=-1$, we also have

$$S^2 = C, \quad C^2 = 1, \quad (ST)^3 = e^{i\theta}C. \tag{52}$$

4. Quasiparticle tunneling algebra

We have seen that the degenerate ground states form a representation of the magnetic translation algebra [Eq. (48)] and the modular transformation algebra [Eqs. (51) and (52)]. The degenerate ground states also form a representation of quasiparticle tunneling algebra. In the following, we will list some important results. The details can be found in Appendix, Sec. 6.

The quasiparticle tunneling algebra is generated by the quasiparticle tunneling operators $A^{(\gamma)}$ and $B^{(\gamma)}$, which are induced by creating a quasiparticle-antiquasiparticle pair $(\gamma,\bar{\gamma})$ and moving the quasiparticle γ in x or y directions around the torus, then annihilate γ with $\bar{\gamma}$. The quasiparticle tunneling operators $A^{(\gamma)}$ and $B^{(\gamma)}$ act within the degenerate ground states on torus and those ground states form a representation of the quasiparticle tunneling algebra. Note that if the quasiparticle γ is described by the occupation distribution $n_{\gamma;l}$ then its antiquasiparticle $\bar{\gamma}$ is described by the distribution $n_{\bar{\gamma};l}=n_{\gamma;l_{\max}-l}$ [see Eqs. (46) and (A18)]. We have $Q_\gamma=-Q_{\bar{\gamma}}$ mod 1.

$A^{(\gamma)}$ and $B^{(\gamma)}$ satisfy the following algebra:

$$A^{(\gamma_2)}A^{(\gamma_1)} = \sum_{\gamma_3} n_{\gamma_1\gamma_2}^{\gamma_3}A^{(\gamma_3)},$$

$$B^{(\gamma_2)}B^{(\gamma_1)}=\sum_{\gamma_3} n^{\gamma_3}_{\gamma_1\gamma_2}B^{(\gamma_3)}, \tag{53}$$

where the fusion coefficients $n^{\gamma_3}_{\gamma_1\gamma_2}$ are non-negative integers that satisfy

$$n^{\gamma'}_{0\gamma}=n^{\gamma'}_{\gamma 0}=\delta_{\gamma\gamma'}, \quad n^{0}_{\gamma\gamma'}=\delta_{\gamma\bar{\gamma}'}. \tag{54}$$

In a basis labeled by the quasiparticle types $|\gamma\rangle$, $A^{(\gamma)}$ and $B^{(\gamma)}$ have a form

$$A^{(\gamma')}|\gamma\rangle=\frac{S_{\bar{\gamma}\gamma'}}{S_{\bar{\gamma}0}}|\gamma\rangle,$$

$$B^{(\gamma_2)}|\gamma_1\rangle=\sum_{\gamma_3} n^{\gamma_3}_{\gamma_1\gamma_2}|\gamma_3\rangle, \tag{55}$$

where $S_{\gamma\gamma'}$ are the matrix elements of the modular transformation S in the basis $|\gamma\rangle$: $S|\gamma\rangle=\Sigma_{\gamma'}|\gamma'\rangle S_{\gamma'\gamma}$.

The magnetic translations T_1 and T_2 still have a form

$$T_1|\gamma\rangle=e^{i2\pi Q_\gamma}|\gamma\rangle, \quad T_2|\gamma\rangle=|\gamma'\rangle \tag{56}$$

in the basis $|\gamma\rangle$, where γ' is the quasiparticle described by the canonical occupation distribution $n_{\gamma';l}=n_{\gamma;(l-1)\%m}$. Due to charge conservation, the fusion coefficients satisfy $n^{\gamma_3}_{\gamma_1\gamma_2}=0$ if $Q_{\gamma_1}+Q_{\gamma_2}\neq Q_{\gamma_3}$. Thus T_1, T_2, $A^{(\gamma)}$, and $B^{(\gamma)}$ satisfy the following algebra:

$$T_1A^{(\gamma)}=A^{(\gamma)}T_1, \quad T_1B^{(\gamma)}=e^{i2\pi Q_\gamma}B^{(\gamma)}T_1,$$

$$T_2A^{(\gamma)}=e^{-i2\pi Q_\gamma}A^{(\gamma)}T_2, \quad T_2B^{(\gamma)}=B^{(\gamma)}T_2. \tag{57}$$

Furthermore T, S, $A^{(\gamma)}$, and $B^{(\gamma)}$ satisfy the following algebra:

$$TA^{(\gamma)}=A^{(\gamma)}T,$$

$$SA^{(\gamma)}=B^{(\gamma)}S. \tag{58}$$

$C=S^2$ has a form

$$C|\gamma\rangle=|\bar{\gamma}\rangle. \tag{59}$$

In addition to the above algebraic relations, the tensor category theory gives us some additional conditions. First, we can obtain a symmetric matrix S^{TC} from S,

$$S^{\text{TC}}_{\gamma\gamma'}=f_\gamma S_{\gamma\gamma'},$$

by choosing proper factors f_γ, where S^{TC} satisfies $S^{\text{TC}}_{0\gamma}=d_\gamma>0$ and $S^{\text{TC}}_{00}=1$. Once we know S, we can use those conditions to fix f_γ. Let us write the diagonal T matrix as

$$T_{\gamma\gamma'}=e^{i2\pi h_\gamma}\delta_{\gamma\gamma'}, \tag{60}$$

where h_γ is the CFT scaling dimension for the quasiparticle γ (see Appendix, Sec. 7). Then S and T should satisfy the following conditions:[7,32]

$$S^{\text{TC}}_{\gamma_1\gamma_2}=\sum_{\gamma_3} n^{\gamma_3}_{\gamma_1\gamma_2}d_{\gamma_3}e^{i2\pi(h_{\gamma_3}-h_{\gamma_1}-h_{\gamma_2})}, \tag{61}$$

where $d_\gamma\equiv S^{\text{TC}}_{0\gamma}=d_{\bar{\gamma}}$, and[32]

$$\exp^{i2\pi\Sigma_{\gamma'}h_{\gamma'}A_{\gamma\gamma'}}=\exp^{i2\pi h_{\gamma}\frac{4}{3}\Sigma_{\gamma'}A_{\gamma\gamma'}} \tag{62}$$

where $A_{\gamma\gamma'}=2n^{\gamma'}_{\gamma\bar{\gamma}}n^{\gamma}_{\gamma\gamma'}+n^{\gamma'}_{\gamma\gamma}n^{\gamma}_{\gamma'\bar{\gamma}}$.

Although we cannot prove it, it is very likely that S is always symmetric, and the factor f_γ is independent of γ. In this case, we have another condition.[32] Let

$$\nu_\gamma=\frac{1}{D^2}\sum_{\gamma'\gamma''} n^{\gamma}_{\gamma'\gamma''}d_{\gamma'}d_{\gamma''}e^{i4\pi(h_{\gamma'}-h_{\gamma''})}, \tag{63}$$

then $\nu_\gamma=0$ if $\gamma\neq\bar{\gamma}$ and $\nu_\gamma=\pm 1$ if $\gamma=\bar{\gamma}$. Here

$$D=\sqrt{\sum_\gamma d^2_\gamma}.$$

E. Summary

In this section, we calculated the charge and the orbital spin of quasiparticle, as well as the ground-state degeneracy from the pattern of zeros $\{S_a\}$ of a FQH states. We also discussed the translation transformations, the modular transformations, and the transformations induced by the quasiparticle tunneling on the degenerate ground states. The algebra of those transformations can help us to determine the quasiparticle statistics, quasiparticle quantum dimensions, and fusion algebra of the quasiparticles. In particular, we can use the algebra [Eqs. (51) and (52)] to determine S and then use Eq. (55) to determine the quasiparticle tunneling operators, $A^{(\gamma)}$ and $B^{(\gamma)}$. The conditions [Eqs. (61)–(63)] can help us to determine quasiparticle scaling dimensions h_γ. The condition that the matrix elements of $B^{(\gamma)}$ must be non-negative integers put further constraint on S.

The main goal in this section is to calculate the data that completely characterize the topological order from the pattern of zeros. At the moment, we do not know what data completely characterize the topological order. But the S and T matrices contain a lot of information about the topological order. If we do find the data that completely characterize the topological order, the same data should also characterize a CFT which will be the theory that describes the edge excitations of the corresponding FQH states. In Ref. 38, some direct relations between the pattern of zeros and CFT are discussed.

V. EXAMPLES

A. Quasiparticles in FQH states

In Ref. 12, many FQH states are characterized and constructed through patterns of zeros. The pattern of zeros in FQH states can be characterized by a $\boldsymbol{S}$-vector $\boldsymbol{S}=(m;S_2,\ldots,S_n)$, a $\boldsymbol{h}$-vector $\boldsymbol{h}=(\frac{m}{n};h^{\text{sc}}_1,\ldots,h^{\text{sc}}_n)$, or an occupation distribution $(n_0\cdots n_{m-1})$. All those data contain information on two important integers n and m. n is the number of electrons in one cluster and m determines the filling fraction $\nu=n/m$.

The FQH states constructed in Ref. 12 include the Z_n parafermion states $\Phi_{n/m;Z_n}$ introduced in Refs. 16 and 39. The patterns of zeros $\{S_a\}$ for those Z_n parafermion states are obtained. The occupation distributions $\{n_l\}$ of those states agree with those obtained in Refs. 34, 35, and 37. Wen and Wang[12] also obtained generalized Z_n parafermion states $\Phi_{n/m;Z_n^{(k)}}$ and their patterns of zeros. $\Phi_{n/m;Z_n^{(k)}}$ has a filling fraction $\nu=n/m$. Many other new FQH states and their patterns of zeros are also obtained in Ref. 12, such as the $\Phi_{n/2n;C_n}$ and the $\Phi_{n/n;C_n/Z_2}$ states.

Once we know the pattern of zeros of a FQH state, we can find all the quasiparticle excitations in such a state by simply finding all $S_{\gamma,a}$ that satisfy Eqs. (22) and (25) [note that $n_{\gamma;l}$ in Eq. (25) are determined from $S_{\gamma;a}$]. From the pattern of zeros that characterizes a quasiparticles, we can find many quantum numbers of that quasiparticle. Here we will summarize those results by just listing the number of quasiparticle types in some FQH states. Then, we will give a more detailed discussion for few simple examples.

For the parafermion states $\Phi_{n/2;Z_n}$ $(m=2)$, we find the numbers of quasiparticle types (NOQT) as follows.

FQH state	$\Phi_{2/2;Z_2}$	$\Phi_{3/2;Z_3}$	$\Phi_{4/2;Z_4}$	$\Phi_{5/2;Z_5}$	$\Phi_{6/2;Z_6}$	$\Phi_{7/2;Z_7}$	$\Phi_{8/2;Z_8}$	$\Phi_{9/2;Z_9}$	$\Phi_{10/2;Z_{10}}$
NOQT	3	4	5	6	7	8	9	10	11

For the parafermion states $\Phi_{n/(2+2n);Z_n}=\Phi_{n/2;Z_n}\prod_{i<j}(z_i-z_j)^2$ $(m=2+2n)$, we find the following table.

FQH state	$\Phi_{2/6;Z_2}$	$\Phi_{3/8;Z_3}$	$\Phi_{4/10;Z_4}$	$\Phi_{5/12;Z_5}$	$\Phi_{6/14;Z_6}$	$\Phi_{7/16;Z_7}$	$\Phi_{8/18;Z_8}$	$\Phi_{9/20;Z_9}$	$\Phi_{10/22;Z_{10}}$
NOQT	9	16	25	36	49	64	81	100	121

For the generalized parafermion states $\Phi_{n/m;Z_n^{(k)}}$, we find the following table.

FQH state	$\Phi_{5/8;Z_5^{(2)}}$	$\Phi_{5/18;Z_5^{(2)}}$	$\Phi_{7/8;Z_7^{(2)}}$	$\Phi_{7/22;Z_7^{(2)}}$	$\Phi_{7/18;Z_7^{(3)}}$	$\Phi_{7/32;Z_7^{(3)}}$	$\Phi_{8/18;Z_8^{(3)}}$	$\Phi_{9/8;Z_9^{(2)}}$	$\Phi_{10/18;Z_{10}^{(3)}}$
NOQT	24	54	32	88	72	128	81	40	99

Here k and n are coprime. For the composite parafermion states $\Phi_{n_1/m_1;Z_{n_1}^{(k_2)}}\Phi_{n_2/m_2;Z_{n_2}^{(k_2)}}$ obtained as products of two parafermion wave functions, we find the following table.

FQH state	$\Phi_{2/2;Z_2}\Phi_{3/2;Z_3}$	$\Phi_{3/2;Z_3}\Phi_{4/2;Z_4}$	$\Phi_{2/2;Z_2}\Phi_{5/2;Z_5}$	$\Phi_{2/2;Z_2}\Phi_{5/8;Z_5^{(2)}}$
NOQT	30	70	63	117

Here n_1 and n_2 are coprime. The filling fractions of the above composite states are $\nu=\frac{n_1n_2}{m_1n_2+m_2n_1}$.

The above results suggest a pattern. For a (generalized) parafermion state $\Phi_{n/m;Z_n^{(k)}}$, we can express its filling fraction as $\nu=n/m=p/q$, where p and q are coprime. Then the number of quasiparticle types is given by NOQT$=qD(n)$, where $D(2)=3$, $D(3)=2$, $D(4)=5$, $D(5)=3$, $D(6)=7$, $D(7)=4$, $D(8)=9$, $D(9)=5$, and $D(10)=11$ or

$$D(n)=n+1 \quad \text{for} \quad n=\text{even},$$

$$D(n)=(n+1)/2 \quad \text{for} \quad n=\text{odd}.$$

Similarly, for a composite parafermion state $\Phi_{n_1/m_1;Z_{n_1}^{(k_1)}}\Phi_{n_2/m_2;Z_{n_2}^{(k_2)}}$, we can express its filling fraction as $\nu=\frac{n_1n_2}{m_1n_2+m_2n_1}=p/q$, where p and q are coprime. Then the number of quasiparticle types is given by NOQT$=qD(n_1)D(n_2)$.

The corresponding CFT of the above (generalized and composite) parafermion states is known. The numbers of the quasiparticle types can also be calculated from the CFT.[38] For the generalized parafermion state $\Phi_{n/m;Z_n^{(k)}}$ the numbers for the quasiparticle types are given by[38]

$$\text{NOQT}=\frac{1}{\nu}\frac{n(n+1)}{2}=\frac{m}{n}\frac{n(n+1)}{2}=\frac{m(n+1)}{2}.$$

For the composite parafermion state $\prod_i\Phi_{n_i/m_i;Z_{n_i}^{(k_i)}}$ the numbers for the quasiparticle types are given by[38]

$$\text{NOQT}=\frac{1}{\nu}\prod_i\frac{n_i(n_i+1)}{2}.$$

Here we require that k_i is not a factor of n_i and $n_1,n_2,n_3,\ldots$ have no common factor. Despite their different forms, the results from CFT agree exactly with the results from the pattern-of-zero approach.

For generalized parafermion states $\Phi_{n/m;Z_n^{(k)}}$, where n and k have a common factor, we have the following table.

FQH state	$\Phi_{4/8;Z_4^{(2)}}$	$\Phi_{6/8;Z_6^{(2)}}$	$\Phi_{6/18;Z_6^{(3)}}$	$\Phi_{8/8;Z_8^{(2)}}$	$\Phi_{8/8;Z_8^{(4)}}$	$\Phi_{9/18;Z_9^{(3)}}$
NOQT	10	20	21	35	36	56

For more general composite parafermion states $\Phi_{n_1/m_1;Z_{n_1}^{(k_2)}}\Phi_{n_2/m_2;Z_{n_2}^{(k_2)}}$, where n_1 and n_2 have a common factor, we have the following table.

FQH state	$\Phi_{2/2;Z_2}\Phi_{2/2;Z_2}$	$\Phi_{2/2;Z_2}\Phi_{4/2;Z_4}$	$\Phi_{3/2;Z_3}\Phi_{3/2;Z_3}$	$\Phi_{4/2;Z_4}\Phi_{4/2;Z_4}$	$\Phi_{5/2;Z_5}\Phi_{5/2;Z_5}$	$\Phi_{5/2;Z_5}\Phi_{5/8;Z_5^{(2)}}$	$\Phi_{5/8;Z_5^{(2)}}\Phi_{5/8;Z_5^{(2)}}$
NOQT	10	42	20	35	56	352	224

For the $\nu=1/2$ states $\Phi_{n/2n;C_n}$, we have the following table.

FQH state	$\Phi_{3/6;C_3}$	$\Phi_{4/8;C_4}$	$\Phi_{5/10;C_5}$	$\Phi_{6/12;C_6}$	$\Phi_{7/14;C_7}$	$\Phi_{8/16;C_8}$	$\Phi_{9/18;C_9}$
NOQT	56	170	352	910	1612	3546	6266

Note that $\Phi_{5/10;C_5}$ and $\Phi_{5/2;Z_5}\Phi_{5/8;Z_5^{(2)}}$ have the same pattern of zeros and may be the same state. For the $\nu=1$ states $\Phi_{n/n;C_n/Z_2}$, we have the following table.

FQH state	$\Phi_{4/4;C_4/Z_2}$	$\Phi_{6/6;C_6/Z_2}$	$\Phi_{8/8;C_8/Z_2}$	$\Phi_{10/10;C_{10}/Z_2}$
NOQT	35	138	171	338

Note that $\Phi_{8/8;Z_8^{(2)}}$, $\Phi_{4/4;C_4/Z_2}$, and $\Phi_{4/2;Z_4}\Phi_{4/2;Z_4}$ have the same pattern of zeros and may be the same state.

For those more general and new FQH states, the corresponding CFT is not identified, even the stability of those FQH states is unclear. If some of those states contain gapless excitations, then the number of quasiparticle types will make no sense for those gapless states. In the following, we will study a few simple examples in more detail.

B. $\nu=1/2$ Laughlin state

For the $\nu=1/2$ Laughlin state, $n=1$ and its pattern of zeros is characterized by

$$\Phi_{1/2}:(m;S_2,\ldots,S_n)=(2;),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right)=\left(\frac{2}{1};0\right),$$

$$n_0\ldots n_{m-1}=10.$$

Solving Eq. (22), we find that there are two types of quasiparticles. Their canonical occupation distributions and other quantum numbers are given by the following table.

$n_{\gamma;l}$	Q_γ	S_γ^{osp}
10	0	0
01	1/2	1/4

Thus T_1, T_2, and T are given by

$$T_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad T_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad T=\begin{pmatrix}1 & 0\\ 0 & t_1\end{pmatrix}.$$

Since $l_{\max}=0$, from Eq. (A18), we find that the conjugate of 10 is $\overline{10}=10$ and the conjugate of 01 is $\overline{01}=01$. Thus $C=1$.

From $TT_2=e^{i\pi n/m}T_2T_1T$ [see Eq. (51)] we find that $t_1=i$. From $ST_1=T_2S$, we find that $S_{10}=S_{00}$ and $S_{01}=-S_{11}$. From $S^2=C=1$, we find that $S_{11}=-S_{00}$ and $S_{00}=\pm\frac{1}{\sqrt{2}}$. Thus

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\quad T=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}.$$

The above implies $(ST)^3=e^{i\pi/4}$. Those modular transformations, S and T, agree with those calculated using the Chern-Simons effective theory[21] and the explicit FQH wave functions.[40] From $T_{11}=i$, we find that the quasiparticle $(n_{\gamma;l})=(01)$ has a scaling dimension $h=1/4$ and a semion statistics.

Let us introduce

$$T_m=e^{i2\pi/24}T,\quad S_m=S.$$

We find that T_m and S_m satisfy

$$S_m=1,\quad (S_mT_m)^3=1.$$

Thus S_m and T_m generate a linear representation of the modular group.

C. Z_2 parafermion state

The $\nu=1$ bosonic Pfaffian state[39] is a Z_2 parafermion state with $n=2$. Its pattern of zeros is described by

$$\Phi_{2/2;Z_2}:(m;S_2,\ldots,S_n)=(2;0),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right)=\left(\frac{2}{2};\frac{1}{2},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0).$$

Solving Eq. (22), we find that there are three types of quasiparticles as shown in the following table.

$n_{\gamma;l}$	Q_γ	S_γ^{osp}
20	0	0
02	1	1/2
11	1/2	1/4

Thus in the $|\gamma\rangle$ basis, T_1, T_2, and T are given by

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{pmatrix}.$$

Since $l_{\max}=0$, from Eq. (A18), we find that $\overline{20}=20$, $\overline{02}=02$, and $\overline{11}=11$. Thus $C=1$.

From $TT_2=e^{i\pi n/m}T_2T_1T$ [see Eq. (51)] we find that $t_1=-1$, but t_2 is undetermined. Using $ST_1=T_2S$, $ST_2=T_1S$, $S^T=S$, and $S^2=1$, we obtain the following possible solutions:

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2}x \\ 1 & 1 & -\sqrt{2}x \\ \sqrt{2}/x & -\sqrt{2}/x & 0 \end{pmatrix}.$$

Those S's satisfy $(ST)^3=t_2$.

Using the above S, we can calculate the fusion coefficients $n^{\gamma_3}_{\gamma_1\gamma_2}$ and $B^{(\gamma)}_{\gamma_1\gamma_2}$ from Eq. (A40). We find that $n^0_{22}=n^1_{22}=1/x^2$. The condition $n^0_{\gamma\bar{\gamma}}=1$ fixes $x=\pm 1$. Thus we have the following two possible solutions:

$$S_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

and

$$S_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

We note that the above S's are already symmetric. Thus those S's can regarded as S^{TC}: $S^{\text{TC}}=2S$.

Now let us calculate $A^{(\gamma)}$ and $B^{(\gamma)}$, $\gamma=0,1,2$, using Eq. (55). For the first solution S_1 we find that

$$(a^{(\gamma')}_{\gamma}) = \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ 1 & -1 & 0 \end{pmatrix},$$

where γ labels rows and γ' labels columns. We have

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$A^{(2)} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that

$$\begin{pmatrix} A^{(0)}A^{(0)} & A^{(0)}A^{(1)} & A^{(0)}A^{(2)} \\ A^{(1)}A^{(0)} & A^{(1)}A^{(1)} & A^{(1)}A^{(2)} \\ A^{(2)}A^{(0)} & A^{(2)}A^{(1)} & A^{(2)}A^{(2)} \end{pmatrix} = \begin{pmatrix} A^{(0)} & A^{(1)} & A^{(2)} \\ A^{(1)} & A^{(0)} & A^{(2)} \\ A^{(2)} & A^{(2)} & A^{(0)}+A^{(1)} \end{pmatrix}.$$

We recover the fusion algebra of the Z_2 parafermion theory. Since S can transform $A^{(\gamma)}$ to $B^{(\gamma)}$ [see Eq. (58)], we also find that

$$B^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We see that $B^{(\gamma)}$ also encode the fusion algebra,

$$\begin{pmatrix} B^{(0)}|0\rangle & B^{(0)}|1\rangle & B^{(0)}|2\rangle \\ B^{(1)}|0\rangle & B^{(1)}|1\rangle & B^{(1)}|2\rangle \\ B^{(2)}|0\rangle & B^{(2)}|1\rangle & B^{(2)}|2\rangle \end{pmatrix} = \begin{pmatrix} |0\rangle & |1\rangle & |2\rangle \\ |1\rangle & |0\rangle & |2\rangle \\ |2\rangle & |2\rangle & |0\rangle+|1\rangle \end{pmatrix},$$

where $|\gamma\rangle$, $\gamma=0,1,2$, are the degenerate ground states.

We note that the third eigenvalue of T, $t_2=e^{i2\pi h_2}$, is still undetermined. Using condition (62), we find that

$$h_2 = \frac{1}{16}\text{mod}\frac{1}{8}.$$

Using condition (63), we further find that

$$1 + e^{i8\pi h_2} = \pm\sqrt{2}e^{i4\pi h_2},$$

which does not put any additional constraint on h_2. The correct value for h_2 is $h_2=1/16$.

Now, let us consider the second solution S_2. For such a solution we have

$$(a^{(\gamma')}_{\gamma}) = \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix}$$

and

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$A^{(2)} = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$B^{(\gamma)}$ remain the same.

The first row of the S^{TC}-matrix is called quantum dimensions of the quasiparticles. For a unitary topological quantum-field theory, all quantum dimensions must be positive real numbers, and moreover ≥ 1. Therefore, the second solution does not give rise to a unitary topological field theory. Based on this reason, we exclude the S_2 solution.

The $\nu=1$ bosonic Z_2 parafermion state $\Phi_{2/2;Z_2}$ has three degenerate ground states on a torus. In the thin torus limit, the three ground states are described by the occupation distributions $\cdots 20202020\cdots$, $\cdots 02020202\cdots$, and $\cdots 11111111\cdots$.

The $\nu=1/2$ fermionic Z_2 parafermion state $\Phi_{2/2;Z_2}\prod(z_i-z_j)$ has six degenerate ground states on a torus. In the thin torus limit, the six ground states are also described by the occupation distributions. Those occupation distributions can be obtained from that of $\Phi_{2/2;Z_2}$ state given above. We note that multiplying the factor $\Pi(z_i-z_j)$ increases the space between every neighboring particles in a distribution by 1. For example, it changes from 11 to 101, 101 to 1001, 2 to 11, 3 to 111, etc. It changes the bosonic distribution $\cdots 202020\cdots$ to a fermionic distribution $\cdots 110011001100\cdots$ and changes the distribution $\cdots 111111\cdots$ to $\cdots 101010101010\cdots$. Including the translated distributions of $\cdots 110011001100\cdots$ and $\cdots 101010101010\cdots$, we find that the fermionic Z_2 parafermion state $\Phi_{2/2;Z_2}\Pi(z_i-z_j)$ has six degenerate ground states described by the distributions

$$\cdots 110011001100\cdots,$$

$$\cdots 011001100110\cdots,$$

$$\cdots 001100110011\cdots,$$

$$\cdots 100110011001\cdots,$$

$$\cdots 101010101010\cdots,$$

$$\cdots 010101010101\cdots.$$

Note that a unit cell contains $m=4$ (or 2) orbitals.

D. Z_3 parafermion state

The $\nu=3/2$ bosonic Z_3 parafermion state has a pattern of zeros described by

$$\Phi_{3/2;Z_3}:(m;S_2,\ldots,S_n)=(2;0,0),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right)=\left(\frac{2}{3};\frac{2}{3},\frac{2}{3},0\right),$$

$$(n_0,\ldots,n_{m-1})=(3,0).$$

Solving Eq. (22), we find that there are four types of quasiparticles as shown in the following table.

$n_{\gamma;l}$	Q_γ	S_γ^{osp}
30	0	0
03	3/2	3/4
12	1	1/2
21	1/2	1/4

Thus in the $|\gamma\rangle$ basis, T_1, T_2, and T are given by

$$T_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\quad T_2=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$T=\begin{pmatrix}1&0&0&0\\0&t_1&0&0\\0&0&t&0\\0&0&0&t_1t_2\end{pmatrix}.$$

Since $l_{\max}=0$, from Eq. (A18), we find that $\overline{30}=30$, $\overline{03}=03$, $\overline{21}=21$, and $\overline{12}=12$. Thus $C=1$.

From $TT_2=e^{i\pi n/m}T_2T_1T$ [see Eq. (51)] we find that $t_1=-i$ and $t_2=t$. Thus

$$T=\begin{pmatrix}1&0&0&0\\0&-i&0&0\\0&0&t&0\\0&0&0&-it\end{pmatrix}.$$

We can rewrite T_1, T_2, and T in direct product form,

$$T_1=\sigma^3\otimes\sigma^0,\quad T_2=\sigma^1\otimes\sigma^0,$$

$$T=\begin{pmatrix}1&0\\0&-i\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&t\end{pmatrix}.$$

Using $ST_1=T_2S$, $ST_2=T_1S$, and $S^2=1$, we find that S must have the following form:

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\begin{pmatrix}\cos(\theta)&\sin(\theta)\\\sin(\theta)&-\cos(\theta)\end{pmatrix}.$$

Note that the above S is already symmetric. Thus we can regard the above S as the S^{TC}.

The direct product form of T_1, T_2, T, and S suggests that the $U(1)$ charge part and the non-Abelian part separate.[33] So let us concentrate on the non-Abelian part,

$$\tilde{T}=\begin{pmatrix}1&0\\0&t\end{pmatrix},\quad \tilde{S}=\begin{pmatrix}\cos(\theta)&\sin(\theta)\\\sin(\theta)&-\cos(\theta)\end{pmatrix}.$$

From $\tilde{S}$, we can calculate $\tilde{A}^{(\gamma)}$ and $\tilde{B}^{(\gamma)}$ [see Eqs. (55) and (58)]. All the matrix elements $\tilde{B}^{(\gamma)}$ must be non-negative integers and $(\tilde{S}\tilde{T})^3\propto 1$.

One way to satisfy those conditions is to let

$$\tan(\theta)=\frac{1+\sqrt{5}}{2}\equiv\varphi.$$

In this case

$$\tilde{T}=\begin{pmatrix}1&0\\0&e^{\pm i4\pi/5}\end{pmatrix},\quad \tilde{S}=\frac{1}{\sqrt{1+\varphi^2}}\begin{pmatrix}1&\varphi\\\varphi&-1\end{pmatrix}\tag{64}$$

and

$$\tilde{A}^{(0)}=1,\quad \tilde{A}^{(1)}=\begin{pmatrix}\varphi&0\\0&1-\varphi\end{pmatrix},$$

$$\tilde{B}^{(0)}=1,\quad \tilde{B}^{(1)}=\begin{pmatrix}0&1\\1&1\end{pmatrix}.$$

Those are the only valid solutions, which are realized by the Fibonacci anions. The solutions with $d_1 < 0$ are excluded as reasoned at the end of the last section, and one of them is the nonunitary Yang-Lee model.

Taking $\gamma_1 = \gamma_2 = 1$ in Eq. (61), we find that

$$-1 = (e^{-i4\pi\tilde{h}_1} + e^{-i2\pi\tilde{h}_1}\varphi)$$

or

$$\cos(2\pi\tilde{h}_1) = -\varphi/2,$$

which gives us

$$\tilde{h}_1 = \pm 2/5 \bmod 1.$$

Since $\tilde{h}_0 = 0$, we see that $e^{i2\pi\tilde{h}_\gamma}$ are the eigenvalues of the $\tilde{T}$ operator [see Eq. (64)]. Let us denote the eigenvalues of the T operator as $e^{i2\pi h_\gamma^T}$. We find, for the two choices of $\tilde{h}_1^T = \pm 2/5$ [see Eq. (64)], the following table.

$n_{\gamma;l}$	h_γ^T	h_γ^T
30	0	0
03	3/4	3/4
12	2/5	3/5
21	3/20	7/20

According to CFT (see Appendix, Sec. 7), the scaling dimension of the quasiparticle operators in the Z_3 parafermion FQH state is $h_\gamma = \frac{Q_\gamma^2}{3} + h_\gamma^{\text{sc}}$. For the quasiparticles 30 and 03, $h_\gamma^{\text{sc}} = 0$ since those are Abelian quasiparticles. For the quasiparticles 12 and 21, $h_\gamma^{\text{sc}} = 1/15$. Thus $h_\gamma = 0$, 3/4, 2/5, and 3/20 for the quasiparticles 03, 30, 12, and 21, respectively, which exactly agree with h_γ^T for the case of $\tilde{h}_1^T = 2/5$. This example demonstrates a way to calculate[32] quasiparticle scaling dimensions from the pattern of zeros.

The $\nu = 3/2$ bosonic Z_3 parafermion state $\Phi_{3/2;Z_3}$ has four degenerate ground states on a torus, described by the occupation distributions

$$\cdots 3030303030 \cdots,$$

$$\cdots 0303030303 \cdots,$$

$$\cdots 2121212121 \cdots,$$

$$\cdots 1212121212 \cdots.$$

The $\nu = 3/5$ fermionic Z_3 parafermion state $\Phi_{3/2;Z_3}\Pi(z_i - z_j)$ has ten degenerate ground states on a torus, described by the occupation distributions

$$\cdots 11100111001110011100111100 \cdots,$$

$$\cdots 01110011100111001110011110 \cdots,$$

$$\cdots 00111001110011100111001111 \cdots,$$

$$\cdots 10011100111001110011100111 \cdots,$$

$$\cdots 11001110011100111001110011 \cdots,$$

$$\cdots 1101011010110101101011010 \cdots,$$

$$\cdots 0110101101011010110101101 \cdots,$$

$$\cdots 1011010110101101011010110 \cdots,$$

$$\cdots 0101101011010110101101011 \cdots,$$

$$\cdots 1010110101101011010110101 \cdots.$$

Note that a unit cell contains $m = 5$ orbitals.

VI. CONCLUSIONS

Through string-net wave functions,[5,6] one can show that nonchiral topological orders can be naturally described and classified by tensor category theory. This raises a question on how to describe and classify the chiral topological order in FQH states. The results in Ref. 12 suggest that the pattern of zeros may provide a way to characterize and classify chiral topological orders in FQH states. In this paper, we see that many topological properties of chiral topological orders can be calculated from the data $\{S_a\}$ that describe the pattern of zeros. In particular, through the algebra of tunneling operators, we see a close connection to tensor category theory. The pattern of zeros provides a link from electron wave functions (the symmetric polynomials) to tensor category theory and the corresponding chiral topological orders.

ACKNOWLEDGMENTS

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APPENDIX

1. Calculation of orbital spin

Let

$$F_{L,J}(\alpha) = \sum_{l=0} (l-J)e^{-\alpha^2 l^2}\delta_{l\%m-L} = \sum_{k=0} (km+L-J)e^{-\alpha^2(km+L)^2}. \tag{A1}$$

We find that, for a canonical occupation distribution (25), we can rewrite Eq. (31) as

$$\theta_B^\gamma = \Omega \sum_{l=0}^{m-1} (n_{\gamma;l} - n_l)F_l(\alpha). \tag{A2}$$

To evaluate $F_{L,J}(\alpha)$, we will use the Euler-Maclaurin formula,

$$\frac{f(0)+f(k_{\max})}{2}+\sum_{k=1}^{k_{\max}} f(k)-\int_0^{k_{\max}} f(x)dx=\sum_{k=1}^{p}\frac{B_{k+1}}{(k+1)!}[f^{(k)}(k_{\max})-f^{(k)}(0)]+R,$$

where B_k are Bernoulli numbers $(B_0,B_1,B_2,B_3,\cdots)=(1,-\frac{1}{2},\frac{1}{6},0,-\frac{1}{30},\cdots)$, and

$$|R|\le\frac{2}{(2\pi)^{2p}}\int_0^n|f^{(2p+1)}(x)|dx.$$

For our case $f(x)=(L+mx-J)e^{-\alpha^2(mx+L)^2}$. If we choose $\alpha\sim 1/J$ and $p=1$, in large J limit, we find $R\sim 1/J$ in large J limit. Therefore

$$F_{L,J}(\alpha)=\int_0^\infty (L+mx-J)e^{-\alpha^2(mx+L)^2}dx+\frac{1}{2}(L-J)-\frac{m}{12}+O(1/J)=\int_0^\infty\frac{L+\frac{x}{\alpha}-J}{m\alpha}e^{-(x+\alpha L)^2}dx+\frac{1}{2}(L-J)-\frac{m}{12}+O(1/J)$$

$$=\int_0^\infty\frac{L+\frac{x}{\alpha}-J}{m\alpha}(1-2\alpha Lx-\alpha^2L^2+2\alpha^2L^2x^2)e^{-x^2}dx+\frac{1}{2}(L-J)-\frac{m}{12}+O(1/J)$$

$$=-\frac{\alpha J\sqrt{\pi}-1}{2\alpha^2 m}+\frac{J}{m}L-\frac{L^2}{2m}+\frac{L-J}{2}-\frac{m}{12}+O(1/J).$$

Since $\Sigma_{l=0}^{m-1}(n_l-n_{\gamma;l})=0$ for a canonical occupation distribution, the terms that do not depend on L do not contribute to the Berry phase. Thus we have

$$\theta_B^\gamma=\Omega\sum_{l=0}^{m-1}(n_{\gamma;l}-n_l)\left(\frac{J}{m}l+\frac{l}{2}-\frac{l^2}{2m}\right). \tag{A3}$$

Compare with Eqs. (26) and (28), we find that the $\frac{J}{m}l$ term exactly reproduces the quasiparticle charge. The terms that do not depend on J give us the orbital spin (32).

Since n_l is a periodic function of l with a period m, we may also view n_l as a periodic function with a period km. If we view n_l as a periodic function with a period km, the orbital spin S_γ^{osp} will be given by

$$\begin{aligned}S_\gamma^{\text{osp}}&=\sum_{l=0}^{km-1}(n_{\gamma;l}-n_l)\left(\frac{l}{2}-\frac{l^2}{2km}\right)\\&=\sum_{j=0}^{k-1}\sum_{l=0}^{m-1}(n_{\gamma;l}-n_l)\left(\frac{jm+l}{2}-\frac{(jm+l)^2}{2km}\right)\\&=\sum_{j=0}^{k-1}\sum_{l=0}^{m-1}(n_{\gamma;l}-n_l)\left(\frac{l}{2}-\frac{jl}{k}-\frac{l^2}{2km}\right)\\&=\sum_{l=0}^{m-1}(n_{\gamma;l}-n_l)\left(\frac{kl}{2}-\frac{k(k-1)l}{2k}-\frac{l^2}{2m}\right)\\&=\sum_{l=0}^{m-1}(n_{\gamma;l}-n_l)\left(\frac{l}{2}-\frac{l^2}{2m}\right),\end{aligned}$$

which is identical to the previous result [Eq. (32)]. Therefore, the formal calculation of S_γ^{osp} using $e^{-\alpha^2l^2}$ regulator [see Eq. (31)] produces a sensible result.

2. Orbital spin of Abelian quasiparticles

If a quasiparticle is described by an occupation distribution $n_{\gamma;l}$ that can be obtained by shifting the occupation distribution n_l for the ground state, then the orbital spin of such a quasiparticle can be calculated reliably without using the formal unreliable approach described above. Such kind of quasiparticles can be created by threading magnetic-flux lines through the FQH liquid and correspond to Abelian quasiparticles.

The occupation distribution n_l for the ground state has some properties that will be important for the following discussion. In addition to the periodic property $n_{l+m}=n_l$, n_l also have a symmetric property

$$n_l=n_{l_{\max}-l},\quad 0\le l\le l_{\max}, \tag{A4}$$

according to numerical experiments, where $l_{\max}=S_n-S_{n-1}$ is the largest l in the first unit cell $0\le l<m$ such that $n_{l_{\max}}>0$ (see Fig. 4). [Equation (A4) implies that $n_0>0$.]

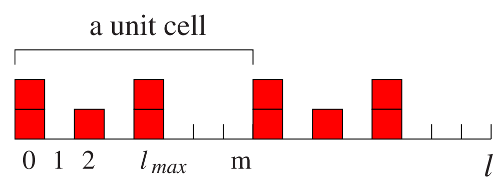


FIG. 4. (Color online) The graphic representation of an occupation distribution $(n_0,\ldots,n_{10})=(2,0,1,0,2,0,0,0)$ for the $\Phi_{5/8;Z_5^{(2)}}$ state with $n=5$ and $m=8$. $l_{\max}=4$ for such a distribution.

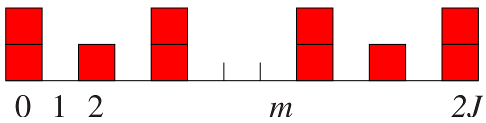


FIG. 5. (Color online) If $N_\phi = 2J = l_{\max} + N_c m$, the symmetric polynomial Φ of $N = N_c n$ variables can be put on the sphere without any defects (or quasiparticles).

On a sphere with $N_\phi = 2J$ flux quanta, there are $N_\phi + 1$ orbitals labeled by $l = 0, 1, \ldots, 2J$ (see Fig. 2). Those orbitals form an angular-momentum J representation of the $O(3)$ rotation. The J_z quantum numbers of those orbitals are given by $m_z = l - J$. If the flux through the sphere is such that $N_\phi = 2J = l_{\max} + (N_c - 1)m$ for an integer N_c, then the occupation n_l can fit into the $2J+1$ orbital in such a way that there is no nontrivial quasiparticle at the north and south poles (see Fig. 5). Such a state has $N = N_c n$ particles. Such an $N_c n$-particle state can fill the sphere without any defect and form a $J^{\text{tot}} = 0$ state. Note that $l_{\max} = l_n = S_n - S_{n-1}$ and $S_{n-1} = \frac{n-2}{n} S_n$ (see Ref. 12), thus $l_{\max} = \frac{2}{n} S_n$. So an $N_c n$-particle state can fill the sphere without any defect if $N_\phi = 2J = \frac{2}{n} S_n + (N_c - 1)m$ which is exactly the condition obtained in Ref. 12.

Let us create a quasiparticle by threading n_ϕ flux lines through the south pole. The total flux quanta become $N_\phi = 2J = \frac{2}{n} S_n + (N_c - 1)m + n_\phi$ and the occupation distribution $n_{\gamma;l}$ for the created quasiparticle is obtained from n_l by shifting the distribution by n_ϕ (see Fig. 6). The occupation distribution is identical to that of the ground-state distribution in Fig. 5 near the north pole ($l = 2J$). Thus the distribution describes a state that has no quasiparticle near the north pole. However, the occupation distribution is different from the ground-state distribution near the south pole ($l = 0$). Therefore, the distribution describes a state with a quasiparticle near the south pole.

The total J_z of the above quasiparticle state is given by

$$
\begin{aligned}
J_z^{\text{tot}} &= \sum_{l=0}^{2J} n_{\gamma;l} m_z = \sum_{l=0}^{2J} n_{\gamma;l}(l - J) = \frac{n_\phi}{2} N \\
&= \frac{n_\phi n}{m}\left(J + \frac{m - l_{\max}}{2} - \frac{n_\phi}{2}\right) \\
&= J Q_\gamma + \frac{m - l_{\max}}{2} Q_\gamma - \frac{m}{2n} Q_\gamma^2,
\end{aligned}
$$

where

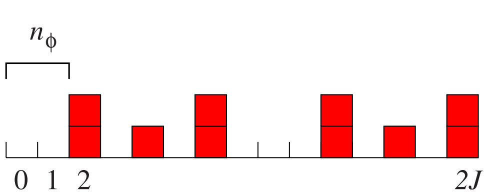


FIG. 6. (Color online) If $N_\phi = 2J = l_{\max} + (N_c - 1)m + n_\phi$, the above occupation distribution $n_{\gamma;l}$ (obtained by shifting the ground-state distribution in Fig. 5) describes a single quasiparticle on the sphere located at the south pole ($l = 0$). There is no quasiparticle at the north pole ($l = 2J$).

$$
Q_\gamma = \frac{n_\phi n}{m} \tag{A5}
$$

is the quasiparticle charge. If we move such quasiparticle along a loop that spans a solid angle Ω, the induced Berry phase θ_B^γ will be

$$
\frac{\theta_B^\gamma}{\Omega} = J_z^{\text{tot}} = J Q_\gamma + \frac{m - l_{\max}}{2} Q_\gamma - \frac{m}{2n} Q_\gamma^2.
$$

Compare with Eq. (30), we find the orbital spin of the quasiparticle to be

$$
S_\gamma^{\text{osp}} = \frac{m - l_{\max}}{2} Q_\gamma - \frac{m}{2n} Q_\gamma^2. \tag{A6}
$$

Let us compare Eq. (A6) with Eq. (32). For the Abelian quasiparticle, its occupation distribution $n_{\gamma;l}$ has a form

$$
n_{\gamma;l} = n_{l - n_\phi}.
$$

In this case, Eq. (32) becomes

$$
\begin{aligned}
S_\gamma^{\text{osp}} &= \sum_{l=0}^{m-1} n_l \left(\frac{l + n_\phi}{2} - \frac{(l + n_\phi)^2}{2m} - \frac{l}{2} + \frac{l^2}{2m} \right) \\
&= \frac{n n_\phi}{2} - \frac{n}{2m} n_\phi^2 - \frac{n_\phi}{m} \sum_{l=0}^{m-1} l n_l.
\end{aligned}
$$

Using Eq. (A4), we find that the above expression agrees with Eq. (A6). This confirms the validity of Eqs. (32) and (33) for the case of Abelian quasiparticles. On the other hand, the validity of Eqs. (32) or (33) for the case of non-Abelian quasiparticle is yet to be confirmed by a more rigorous calculation.

3. Translations of orbitals on torus

To show $T_2 \phi^{(l)} = \phi^{(l+1)}$, we note that

$$
\begin{aligned}
T_2 \phi^{(l)}(X^1, X^2) = {}& e^{i2\pi X^1} f^{(l)} \\
& \times \left[X^1 + \tau \left(X^2 + \frac{1}{N_\phi} \right) \right] e^{i\pi N_\phi \tau [X^2 + (1/N_\phi)]^2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
f^{(l)}\left(z + \frac{\tau}{N_\phi}\right) &= \sum_k e^{i(\pi\tau/N_\phi)(N_\phi k + l)^2 + i2\pi(N_\phi k + l)[z + (\tau/N_\phi)]} \\
&= e^{-i(\pi\tau/N_\phi)} \sum_k e^{i(\pi\tau/N_\phi)(N_\phi k + l + 1)^2 + i2\pi(N_\phi k + l)z} \\
&= e^{-i(\pi\tau/N_\phi)} e^{-i2\pi z} f^{(l+1)}(z),
\end{aligned}
$$

we have

$$
\begin{aligned}
T_2 \phi^{(l)}(X^1, X^2) = {}& e^{i2\pi X^1 - i(\pi\tau/N_\phi) - i2\pi z} f^{(l+1)}(X^1 \\
& + \tau X^2) e^{i\pi N_\phi \tau [X^2 + (1/N_\phi)]^2} = \phi^{(l+1)}(X^1, X^2).
\end{aligned}
$$

4. Non-Abelian Berry's phase and modular transformation

The wave functions $\Phi_{\{n_{\gamma;l+l_s}\}}$ form a basis of the degenerate ground states. As we change the mass matrix or τ, we obtain a family of basis parametrized by τ. The family of basis can give rise to non-Abelian Berry's phase[41] which contains a lot of information on topological order in the FQH state. In the following, we will discuss such non-Abelian Berry's phase in a general setting. We will use γ to label the degenerate ground states.

To find the non-Abelian Berry's phase, let us first define parallel transportation of a basis. Consider a path $\tau(s)$ that deforms the inverse-mass matrix $g(\tau_1)$ to $g(\tau_2)$: $\tau_1=\tau(0)$ and $\tau_2=\tau(1)$. Assume that for each inverse-mass matrix $g[\tau(s)]$, the many-electron Hamiltonian on torus $(X^1,X^2)\sim(X^1+1,X^2)\sim(X^1,X^2+1)$ has N_q-fold degenerate ground states $|\gamma;s\rangle$, $\gamma=0,1,\ldots,N_q-1$, and a finite-energy gap for excitations above the ground states. We can always choose a basis $|\gamma;s\rangle$ for the ground states such that the basis for different s satisfies

$$\langle\gamma';s|\frac{d}{ds}|\gamma;s\rangle=0.$$

Such a choice of basis $|\gamma;s\rangle$ defines a parallel transportation from the bases for inverse-mass matrix $g(\tau_1)$ to that for inverse-mass matrix $g(\tau_2)$ along the path $\tau(s)$.

In general, the parallel transportation is path dependent. If we choose another path $\tau'(s)$ that connect τ_1 and τ_2, the parallel transportation of the same basis for inverse-mass matrix $g(\tau_1)$, $|\gamma;s=0\rangle=|\gamma;s=0\rangle'$, may result in a different basis for inverse-mass matrix $g(\tau_2)$, $|\gamma;s=1\rangle\neq|\gamma;s=1\rangle'$. The different bases are related by an invertible transformation. Such a path-dependent invertible transformation is the non-Abelian Berry's phase.[41]

However, for the degenerate ground states of a topologically ordered state (including a FQH state), the parallel transportation has a special property that, up to a total phase, it is path independent (in the thermal dynamical limit). The parallel transportations along different paths connecting τ_1 and τ_2 will change a basis for inverse-mass matrix $g(\tau_1)$ to the same basis for inverse-mass matrix $\tau(\tau_2)$ up to an overall phase: $|\gamma;s=1\rangle=e^{i\phi}|\gamma;s=1\rangle'$. In particular, if we deform an inverse-mass matrix through a loop into itself $[ie\tau(0)=\tau(1)]$, the basis $|\gamma;0\rangle$ will parallel transport into $|\gamma;1\rangle=e^{i\varphi}|\gamma;0\rangle$. Thus, non-Abelian Berry's phases for the degenerate states of a topologically ordered state are only path-dependent Abelian phases $e^{i\varphi}$ which do not contain much information of topological order.

However, there is a class of special paths which give rise to nontrivial non-Abelian Berry's phases. First we note that the torus $(X^1,X^2)\sim(X^1+1,X^2)\sim(X^1,X^2+1)$ can be parametrized by another set of coordinates,

$$\begin{pmatrix}X'^1\\X'^2\end{pmatrix}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}X^1\\X^2\end{pmatrix},\quad\begin{pmatrix}X^1\\X^2\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}X'^1\\X'^2\end{pmatrix},\tag{A7}$$

where $a,b,c,d\in Z$, $ad-bc=1$. The above can be rewritten in vector form,

$$\boldsymbol{X}'=M^{-1}\boldsymbol{X},\quad\boldsymbol{X}=M\boldsymbol{X}',\quad M\in SL(2,Z).\tag{A8}$$

(X'^1,X'^2) has the same periodicity condition $(X'^1,X'^2)\sim(X'^1+1,X'^2)\sim(X'^1,X'^2+1)$ as that for (X^1,X^2). We note that

$$\begin{pmatrix}\partial_{X^1}\\\partial_{X^2}\end{pmatrix}=\begin{pmatrix}d&-c\\-b&a\end{pmatrix}\begin{pmatrix}\partial_{X'^1}\\\partial_{X'^2}\end{pmatrix},\quad\begin{pmatrix}\partial_{X'^1}\\\partial_{X'^2}\end{pmatrix}=\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}\partial_{X^1}\\\partial_{X^2}\end{pmatrix}.$$

The inverse-mass matrix in the (X^1,X^2) coordinate, $g(\tau)$, is changed to

$$g'=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}g(\tau)\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$$

in the (X'^1,X'^2) coordinate. From Eq. (36), we find that

$$g'=\begin{pmatrix}\tau_y'+\frac{\tau_x'^2}{\tau_y'}&-\frac{\tau_x'}{\tau_y'}\\-\frac{\tau_x'}{\tau_y'}&\frac{1}{\tau_y'}\end{pmatrix}=g(\tau'),\tag{A9}$$

with

$$\tau'=\frac{b+d\tau}{a+c\tau}.\tag{A10}$$

The above transformation $\tau\to\tau'$ is the modular transformation. We see that if τ and τ' are related by the modular transformation, then two inverse-mass matrices $g(\tau)$ and $g(\tau')$ will actually describe the same system (up to a coordinate transformation).

Let us assume that the path $\tau(s)$ connects two τ's related by a modular transformation,

$$M=\begin{pmatrix}a&b\\c&d\end{pmatrix},$$

$\tau(1)=\frac{b+d\tau(0)}{a+c\tau(0)}$. We will denote $\tau(0)=\tau$ and $\tau(1)=\tau'$. The parallel transportation of the basis $|\gamma;\tau\rangle$ for inverse-mass matrix $g[\tau(0)]$ gives us a basis $|\gamma;\tau'\rangle$ for inverse-mass matrix $g[\tau(1)]$. Since $\tau=\tau(0)$ and $\tau'=\tau(1)$ are related by a modular transformation, $g[\tau(0)]$ and $g[\tau(1)]$ actually describe the same system. The two bases $|\gamma;\tau\rangle$ and $|\gamma;\tau'\rangle$ are actually two bases of same space of the degenerate ground states. Thus there is an invertible matrix that relates the two bases,

$$|\gamma;\tau'\rangle=U(M)|\gamma;\tau\rangle,$$

$$U_{\gamma\gamma'}(M)=\langle\gamma;\tau|\gamma';\tau'\rangle=\langle\gamma;\tau|U(M)|\gamma';\tau\rangle.\tag{A11}$$

Such an invertible matrix is the non-Abelian Berry's phase for the path $\tau(s)$. Except for its overall phase (which is path dependent), the invertible matrix U is a function of the modular transformation M. In fact, the invertible matrix U forms a projective representation of the modular transformation. The projective representation of the modular transformation contains a lot of information of the underlying topological order.

Let us examine $\langle\gamma;\tau|\gamma';\tau\rangle$ in Eq. (A11) more carefully. Let $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau]$ be ground-state wave functions for inverse-mass matrix $g[\tau]$ and $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau']$ be ground-state wave

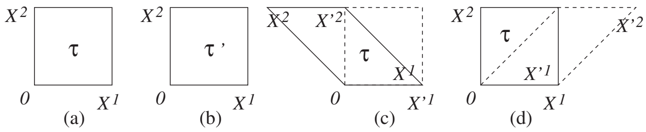


FIG. 7. (a) A system described by an inverse-mass matrix $g(\tau)$. (b) A system described by an inverse-mass matrix $g(\tau')$ where τ and τ' are related by a modular transformation M_T. (c) The system in (b) is the same as the system in (a) if we make a change in coordinates. (d) The system in (c) is drawn differently.

functions for inverse-mass matrix $g[\tau']$. Here $\boldsymbol{X}_i=(X_i^1,X_i^2)$ are the coordinates of the ith electron. Since τ and τ' are related by a modular transformation, $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau]$ and $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau']$ are ground-state wave function of the same system. However, we cannot directly compare $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau]$ and $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau']$ and calculate the inner product between the two wave functions as

$$(\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau(0)], \quad \Phi_{\gamma'}[\{\boldsymbol{X}_i\}|\tau']).$$

The wave function $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau']$ for inverse-mass matrix $g[\tau']$ can be viewed as the ground-state wave function for inverse-mass matrix $g[\tau]$ only after a coordinate transformation (see Fig. 7). Let us rename $\boldsymbol{X}$ to $\boldsymbol{X}'$ and rewrite $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau']$ as $\Phi_\gamma(\{\boldsymbol{X}_i'\}|\tau')$. Since the coordinate transformation (A8) changes from τ to τ', we see that we should really compare $\Phi_\gamma(\{\boldsymbol{X}_i'\}|\tau')=\Phi_\gamma(\{M^{-1}\boldsymbol{X}_i\}|\tau')$ with $\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau)$. But even $\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau)$ and $\Phi_\gamma(\{M^{-1}\boldsymbol{X}_i\}|\tau')$ cannot be directly compared. This is because the coordinate transformation (A7) changes the gauge potential (37) to another gauge equivalent form. We need to perform a $U(1)$ gauge transformation $U_G(M)$ to transform the changed gauge potential back to its original form [Eq. (37)]. So only $\Phi_\gamma[\{\boldsymbol{X}_i\}|\tau]$ and $U_G\Phi_\gamma[\{M^{-1}\boldsymbol{X}_i\}|\tau']$ can be directly compared. Therefore, we have

$$U_{\gamma\gamma'}(M) = \{\Phi_\gamma[(\boldsymbol{X}_i)|\tau], U_G(M)\Phi_{\gamma'}[(M^{-1}\boldsymbol{X}_i)|\tau']\} \tag{A12}$$

which is Eq. (A11) in wave function form. Note that $\tau'=\frac{M_{12}+M_{22}\tau}{M_{11}+M_{21}\tau}$.

Let us calculate the gauge transformation $U_G(M)$. We note that $\Phi_\gamma(\{\boldsymbol{X}_i'\}|\tau')$ is the ground state of

$$H' = -\sum_k \frac{1}{2}\sum_{i,j=1,2}\left(\frac{\partial}{\partial X_k'^i} - iA_i'\right)g_{ij}'\left(\frac{\partial}{\partial X_k'^j} - iA_j'\right),$$

where $k=1,\ldots,N$ labels the different electrons. In terms of X^i [see Eq. (A7)], H' has a form

$$H' = -\sum_k \frac{1}{2}\sum_{i,j=1,2}\left(\frac{\partial}{\partial X_k^i} - i\tilde{A}_i\right)g_{ij}\left(\frac{\partial}{\partial X_k^j} - i\tilde{A}_j\right),$$

where

$$\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} = \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}A_1'\\ A_2'\end{pmatrix}, \quad \begin{pmatrix}A_1'\\ A_2'\end{pmatrix} = \begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix}.$$

Since $(A_1',A_2')=(-2\pi N_\phi X'^2,0)$, we find that

$$\begin{aligned}(\tilde{A}_1,\tilde{A}_2) &= (-2\pi N_\phi X'^2 d, 2\pi N_\phi X'^2 b)\\ &= [-2\pi N_\phi(-cX^1+aX^2)d, 2\pi N_\phi(-cX^1+aX^2)b]\end{aligned}$$

$U_G(M)$ will change H' to H,

$$\begin{aligned}U_G(M)H'U_G^\dagger(M) &= H\\ &= -\sum_k \frac{1}{2}\sum_{i,j=1,2}\left(\frac{\partial}{\partial X_k^i} - iA_i\right)g_{ij}\left(\frac{\partial}{\partial X_k^j} - iA_j\right)\end{aligned}$$

with $(A_1,A_2)=(-2\pi N_\phi X^2,0)$. We find that

$$U_G(M) = \prod_k u_G(M;\boldsymbol{X}_k),$$

$$u_G(M;\boldsymbol{X}) = e^{i2\pi N_\phi[bcX^1X^2-(cd/2)(X^1)^2-(ab/2)(X^2)^2]}. \tag{A13}$$

Equation (A12) can also be rewritten as a transformation on the wave function $\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau)$,

$$\begin{aligned}U(M)\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) &= U_G(M)\Phi_\gamma(\{\boldsymbol{X}_i'\}|\tau')\\ &= U_G(M)\Phi_\gamma(\{M^{-1}\boldsymbol{X}_i\}|\tau')\\ &= \Phi_{\gamma'}(\{\boldsymbol{X}_i\}|\tau)U_{\gamma'\gamma}(M),\end{aligned} \tag{A14}$$

where $\tau'=\frac{M_{12}+M_{22}\tau}{M_{11}+M_{21}\tau}$. We see that the action of the operator $U(M)$ on a wave function $\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau)$ is to replace $\boldsymbol{X}_i$ by $\boldsymbol{X}_i'=M^{-1}\boldsymbol{X}_i$, replace τ by $\tau'=\frac{M_{12}+M_{22}\tau}{M_{11}+M_{21}\tau}$, and then multiply a phase factor $U_G(M)$ given in Eq. (A13). Thus Eq. (A14) defines a way how modular transformations act on functions. We find that

$$\begin{aligned}U(M')\{U(M)\Phi_\gamma[(\boldsymbol{X}_i)|\tau]\} &= U(M')\{U_G(M)\Phi_\gamma[(M^{-1}\boldsymbol{X}_i)|\tau']\}\\ &= U_G(M'M)\Phi_\gamma[(M^{-1}M'^{-1}\boldsymbol{X}_i)|\tau'']\\ &= U_G(M'M)\Phi_\gamma\{[(M'M)^{-1}\boldsymbol{X}_i]|\tau''\}\\ &= U(M'M)\Phi_\gamma[(\boldsymbol{X}_i)|\tau].\end{aligned}$$

Here $\tau'=\frac{M_{12}+M_{22}\tau}{M_{11}+M_{21}\tau}$ and $\tau''=\frac{M_{12}+M_{22}\tilde{\tau}}{M_{11}+M_{21}\tilde{\tau}}$ with $\tilde{\tau}=\frac{M_{12}'+M_{22}'\tau}{M_{11}'+M_{21}'\tau}$. Thus $U(M')U(M)=U(M'M)$. So $U(M)$ form a faithful representation of modular transformations $SL(2,Z)$.

To summarize, there are two kinds of deformation loops $\tau(s)$. If $\tau(0)=\tau(1)$, the deformation loop is contractible [i.e., we can deform the loop to a point or in other words we can continuously deform the function $\tau(s)$ to a constant function $\tau(s)=\tau(0)=\tau(1)$]. For a contractible loop, the associated non-Abelian Berry's phase is actually a $U(1)$ phase, $U_{\gamma\gamma'}=e^{i\varphi}\delta_{\gamma\gamma'}$, where φ is path dependent. If $\tau(0)$ and $\tau(1)$ are related by a modular transformation, the deformation loop is noncontractible. Then the associated non-Abelian Berry's phase is nontrivial. If two noncontractible loops can be deformed into each other continuously, then the two loops only differ by a contractible loop. The associated non-Abelian Berry's phases will only differ by an overall $U(1)$ phase. Thus, up to an overall $U(1)$ phase, the non-Abelian Berry's phases $U_{\gamma\gamma'}$ of a topologically ordered state are determined by the modular transformation $\tau\to\tau'=\frac{a\tau+b}{c\tau+d}$. We also show that we can use the parallel transportation to defined a system of basis $\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau)$ for all inverse-mass matrices labeled by τ. By considering the relation of those basis for two τ's related by a modular transformation, we can even obtain a faithful representation of the modular transformation $SL(2,Z)$.

5. Algebra of modular transformations and translations

The translation $T_{\boldsymbol{d}}$ and modular transformation $U(M)$ all act within the space of degenerate ground states. There is an algebraic relation between those operators. From Eq. (A14), we see that

$$\begin{aligned} U(M)T_{\boldsymbol{d}}\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) &= U(M)\Phi_\gamma(\{\boldsymbol{X}_i+\boldsymbol{d}\}|\tau) \\ &= U_G\Phi_\gamma(\{M^{-1}\boldsymbol{X}_i+\boldsymbol{d}\}|\tau') \\ &= U_G\Phi_\gamma(\{M^{-1}(\boldsymbol{X}_i+M\boldsymbol{d})\}|\tau'). \end{aligned}$$

Therefore

$$e^{i\theta}U(M)T_{\boldsymbol{d}} = T_{M\boldsymbol{d}}U(M). \tag{A15}$$

Let us determine the possible phase factor $e^{i\theta}$ for some special cases. Consider the modular transformation $\tau\to\tau'=\tau+1$ generated by

$$M_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We first calculate

$$\begin{aligned} U(M_T)T_1\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) &= U(M_T)\Phi_\gamma(\{\boldsymbol{X}_i+\boldsymbol{d}_1\}|\tau) \\ &= e^{-i\pi N_\phi \sum_i (X_i^2)^2}\Phi_\gamma(\{M_T^{-1}\boldsymbol{X}_i+\boldsymbol{d}_1\}|\tau'), \end{aligned}$$

where $\boldsymbol{d}_1=(\frac{1}{N_\phi},0)$. We note that

$$M_T\boldsymbol{d}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{N_\phi} \\ 0 \end{pmatrix} = \boldsymbol{d}_1.$$

Thus we next calculate

$$\begin{aligned} &T_1U(M_T)\Phi_\gamma[(\boldsymbol{X}_i)|\tau] \\ &= T_1\exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma[(M_T^{-1}\boldsymbol{X}_i)|\tau'] \\ &= \exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma\{[M_T^{-1}(\boldsymbol{X}_i+\boldsymbol{d}_1)]|\tau'\} \\ &= \exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma[(M_T^{-1}\boldsymbol{X}_i+\boldsymbol{d}_1)|\tau']. \end{aligned}$$

Therefore,

$$U(M_T)T_1 = T_1U(M_T).$$

To obtain the algebra between $U(M_T)$ and T_2, we first calculate

$$\begin{aligned} &U(M_T)T_2\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) \\ &= U(M_T)\exp\left[i2\pi\sum_i X_i^1\right]\Phi_\gamma(\{\boldsymbol{X}_i+\boldsymbol{d}_2\}|\tau) \\ &= \exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\exp\left[i2\pi\sum_i (X_i^1-X_i^2)\right] \\ &\times\Phi_\gamma(\{M_T^{-1}\boldsymbol{X}_i+\boldsymbol{d}_2\}|\tau'), \end{aligned}$$

where $\boldsymbol{d}_2=(0,\frac{1}{N_\phi})$. We note that

$$M_T\boldsymbol{d}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{N_\phi} \end{pmatrix} = \boldsymbol{d}_1+\boldsymbol{d}_2.$$

Thus we next calculate

$$\begin{aligned} T_2T_1U(M_T)\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) &= T_2T_1\exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma(\{M_T^{-1}\boldsymbol{X}_i\}|\tau') = T_2\exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma\{[M_T^{-1}(\boldsymbol{X}_i+\boldsymbol{d}_1)]|\tau'\} \\ &= \exp\left[i2\pi\sum_i X_i^1\right]\exp\left[-i\pi N_\phi\sum_i \left(X_i^2+\tfrac{1}{N_\phi}\right)^2\right]\Phi_\gamma\{[M_T^{-1}(\boldsymbol{X}_i+\boldsymbol{d}_1+\boldsymbol{d}_2)]|\tau'\} \\ &= e^{-i\pi N/N_\phi}\exp\left[i2\pi\sum_i (X_i^1-X_i^2)\right]\exp\left[-i\pi N_\phi\sum_i (X_i^2)^2\right]\Phi_\gamma\{[\ M_T^{-1}\boldsymbol{X}_i+\boldsymbol{d}_2)]|\tau'\}, \end{aligned}$$

We see that

$$U(M_T)T_2 = e^{i\pi(n/m)}T_2T_1U(M_T).$$

Next we consider the modular transformation $\tau\to\tau'=-1/\tau$ generated by

$$M_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

From

$$U(M_S)\Phi_\gamma(\{\boldsymbol{X}_i\}|\tau) = \exp\left[-i2\pi N_\phi\sum_i X_i^1X_i^2\right]\Phi_\gamma(\{M_S^{-1}\boldsymbol{X}_i\}|\tau'),$$

we find

$$U(M_S)U(M_S)\Phi_\gamma(\{X_i\}|\tau) = U(M_S)\exp\left[-i2\pi N_\phi \sum_i X_i^1 X_i^2\right]\Phi_\gamma(\{M_S^{-1}X_i\}|\tau') = \exp\left[-i2\pi N_\phi \sum_i X_i^1 X_i^2\right]\exp\left[-i2\pi N_\phi \sum_I X_i^2(-X_i^1)\right] \times \Phi_\gamma(\{-X_i\}|\tau) = \Phi_\gamma(\{-X_i\}|\tau).$$

We see that $U(M_S)U(M_S)=U(-1)$ generates to transformation $\boldsymbol{X}\to-\boldsymbol{X}$. We can show that

$$U(-1)T_1U^{-1}(-1) = T_1^{-1}, \quad U(-1)T_2U^{-1}(-1) = T_2^{-1}. \tag{A16}$$

Since the wave function of an orbital satisfies

$$\phi^{(l)}(-X^1,-X^2) = \phi^{(N_\phi-l)}(X^1,X^2) = \phi^{(-l)}(X^1,X^2),$$

we find

$$U(M_S)U(M_S)\Phi_\gamma(\{X_i\}|\tau) = U(-1)\Phi_\gamma(\{X_i\}|\tau) = \Phi_{\bar\gamma}(\{X_i\}|\tau), \tag{A17}$$

where $\bar\gamma$ corresponds to the occupation distribution,

$$n_{\bar\gamma;l+l_s} = n_{\gamma;-l+l_s}.$$

Since $n_{\gamma,l}$ and $n_{\bar\gamma,l}$ are periodic with a period of m, the above can be rewritten as

$$n_{\bar\gamma;l} = n_{\gamma;(l_{\max}-l)\%m}. \tag{A18}$$

$\bar\gamma$ corresponds to the antiquasiparticle of γ. Thus we find that

$$[U(M_S)]^2 = C, \quad C\Phi_\gamma = \Phi_{\bar\gamma},$$

where C is the quasiparticle conjugation operator. Clearly $C^2=1$.

Let us first calculate

$$U(M_S)T_1\Phi_\gamma(\{X_i\}|\tau) = U(M_S)\Phi_\gamma(\{M_S^{-1}X_i+d_1\}|\tau) = \exp\left[-i2\pi N_\phi \sum_i X_i^1X_i^2\right] \times \Phi_\gamma(\{M_S^{-1}X_i+d_1\}|\tau').$$

Since $M_S d_1 = d_2$, we next consider

$$T_2U(M_S)\Phi_\gamma(\{X_i\}|\tau) = T_2\exp\left[-i2\pi N_\phi \sum_i X_i^1X_i^2\right]\Phi_\gamma(\{M_S^{-1}X_i\}|\tau') = \exp\left[i2\pi \sum_i X_i^1\right]\exp\left[-i2\pi N_\phi \sum_i X_i^1\left(X_i^2+\tfrac{1}{N_\phi}\right)\right]\Phi_\gamma\{[M_S^{-1}(X_i+d_2)]|\tau'\} = \exp\left[-i2\pi N_\phi \sum_i X_i^1X_i^2\right]\Phi_\gamma\{[M_S^{-1}(X_i+d_2)]|\tau'\}.$$

We see that

$$U(M_S)T_1 = T_2U(M_S),$$

$$U(M_S)T_2 = T_1^{-1}U(M_S),$$

where we have used Eq. (A16). Let us introduce

$$T = e^{i\theta}U(M_T), \quad S = U(M_S),$$

where the value of θ will be chosen to make $T_{00}=1$. We find that

$$TT_1 = T_1T, \quad TT_2 = e^{i\pi(n/m)}T_2T_1T,$$

$$ST_1 = T_2S, \quad ST_2 = CT_1C^{-1}S. \tag{A19}$$

6. Quasiparticle tunneling algebra

a. Quasiparticle tunneling around a torus

To obtain the quasiparticle tunneling algebra, it is useful to consider the following tunneling process: (a) we first create a quasiparticle γ_1 and its antiquasiparticle $\bar\gamma_1$, then (b) move the quasiparticle around the torus to wrap the torus n_1 times in the X^1 direction and n_2 times in the X^2 direction, and last (c) we annihilate γ_1 and $\bar\gamma_1$. The quasiparticle-pair-creation process in step (a) is represented by an operator that map no-quasiparticle-particle states to two-quasiparticle-particle states. The quasiparticle transport process in step (b) is represented by an operator that map two-quasiparticle-particle states to two-quasiparticle-particle states. The quasiparticle-pair annihilation process in step (c) is represented by an operator that map two-quasiparticle-particle states to no-quasiparticle-particle states. The whole tunneling process induces a transformation between the degenerate ground states on the torus,

$$|\gamma\rangle \to W^{(\gamma_1)}_{(n_1,n_2)}|\gamma\rangle = |\gamma'\rangle (W^{(\gamma_1)}_{(n_1,n_2)})_{\gamma'\gamma}.$$

For Abelian FQH states, $W^{(\gamma_1)}_{(n_1,n_2)}$ is always an invertible transformation. But for non-Abelian FQH states, $W^{(\gamma_1)}_{(n_1,n_2)}$ may *not* be invertible. This is because when we create the quasiparticle-antiquasiparticle pair in step (a), the pair is in such a state that they fuse into the identity channel. But after

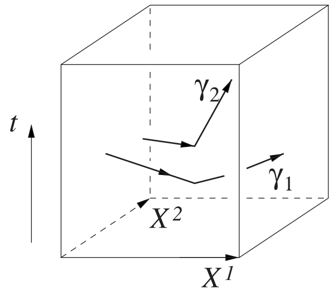


FIG. 8. The tunneling processes $A^{(\gamma_1)}$ and $B^{(\gamma_2)}$.

wrapping the quasiparticle around the torus, the fusion channel may change and hence the pair may not be able to annihilate into ground in step (c). In other words, the annihilate process in step (c) represents a projection into the subspace spanned by the degenerate ground states.

Let (see Fig. 8) $A^{(\gamma_1)}=W^{(\gamma_1)}_{(1,0)}$ and $B^{(\gamma_1)}=W^{(\gamma_1)}_{(0,1)}$. A combination of two tunneling processes in the X^1 direction, $A^{(\gamma_1)}$ and then $A^{(\gamma_2)}$, induces a transformation $A^{(\gamma_2)}A^{(\gamma_1)}$ on the degenerate ground states. A combination of the same two tunneling processes but with a different time order, $A^{(\gamma_2)}$ and then $A^{(\gamma_1)}$, induces a transformation $A^{(\gamma_1)}A^{(\gamma_2)}$ on the degenerate ground states. We note that the two tunneling paths with different time orders can be deformed into each other smoothly. So they only differ by local perturbations. Due to the topological stability of the degenerate ground states,[3] local perturbations cannot change the degenerate ground states. Therefore $A^{(\gamma_1)}$ and $A^{(\gamma_2)}$ commute, and similarly $B^{(\gamma_1)}$ and $B^{(\gamma_2)}$ commute too. We see that $A^{(\gamma)}$'s can be simultaneously diagonalized. Similarly, $B^{(\gamma)}$'s can also be simultaneously diagonalized. Due to the 90° rotation symmetry, $A^{(\gamma)}$ and $B^{(\gamma)}$ have the same set of eigenvalues. But since $A^{(\gamma)}$ and $B^{(\gamma)}$ in general do not commute, we in general cannot simultaneously diagonalize $A^{(\gamma)}$ and $B^{(\gamma)}$.

The basis $\Phi_\gamma=\Phi_{\{n_{\gamma;l+l_s}\}}$ described by the occupation distribution $n_{\gamma;l+l_s}$ on the orbitals $\phi^{(l)}$ is a natural basis in which $A^{(\gamma_1)}$ is diagonal. This is because the tunneling process $A^{(\gamma_1)}$ does not move quasiparticle in the X^2 direction and hence does not modify the occupation distribution $n_{\gamma;l+l_s}$ on the orbitals $\phi^{(l)}$. On the other hand, $B^{(\gamma_1)}$ does move quasiparticle in the X^2 direction and hence shifts the occupation distribution $n_{\gamma;l+l_s}$ that characterizes the ground states. Therefore $B^{(\gamma_1)}$ is not diagonal in the Φ_γ basis. In particular, when acted on the state $\Phi_0\equiv\Phi_{\{n_{l+l_s}\}}$ that corresponds to the trivial quasiparticle, $B^{(\gamma_1)}$ produces the state Φ_{γ_1} that corresponds the quasiparticle γ_1,

$$B^{(\gamma_1)}\Phi_0=b_{\gamma_1}\Phi_{\gamma_1}, \tag{A20}$$

where b_{γ_1} is a nonzero factor. (Note that $\gamma=0,1,\ldots,N_q-1$ where $\gamma=0$ corresponds to the trivial quasiparticle and N_q is the number of quasiparticle types.)

b. Tensor category structure in quasiparticle tunneling operators

$W^{(\gamma)}_{(n_1,n_2)}$ is just a special kind of quasiparticle tunneling. In general, we can create many pairs of quasiparticles, move them around each other, combine and/or split quasiparticles,

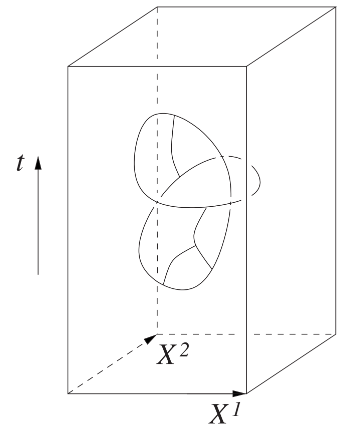
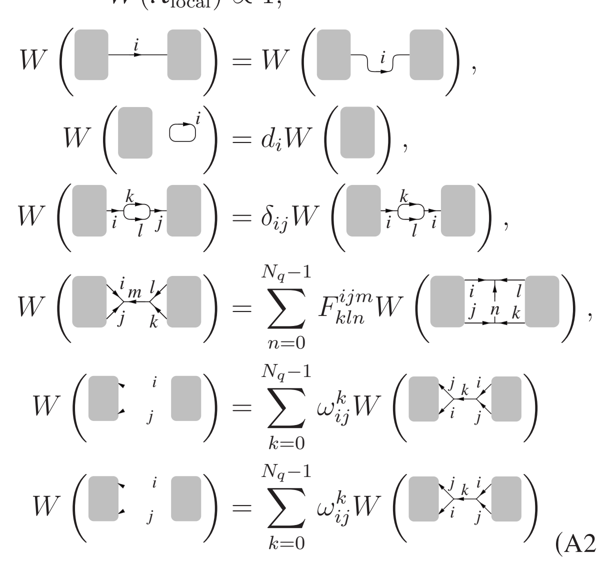


FIG. 9. A general quasiparticle tunneling process.

and then annihilate all of them (see Fig. 8). In addition to the quasiparticle-pair-creation process represented by a mapping from no-quasiparticle-particle states to two-quasiparticle-particle states and the quasiparticle-pair annihilation process represented by a mapping from two-quasiparticle-particle states to no-quasiparticle-particle states, the more general tunneling process also contains the quasiparticle splitting process represented by a mapping from one-quasiparticle-particle states to two-quasiparticle-particle states and the quasiparticle fusion process represented by a mapping from two-quasiparticle-particle states to one-quasiparticle-particle states. We will use $W(\mathcal{X})$ to represent the action of the whole tunneling process on the degenerate ground state where $\mathcal{X}$ represents the tunneling path (see Fig. 9). $W^{(\gamma_1)}_{(n_1,n_2)}$ discussed above is just a special case of $W(\mathcal{X})$.

Due to the topological stability of the degenerate ground states $W(\mathcal{X})$ can have some very nice algebraic properties. However, in order for $W(\mathcal{X})$ to have the nice algebraic structure we need to choose the operators that represent the quasiparticle-pair-creation or annihilation and quasiparticle splitting or fusion processes properly. We conjecture that after making those choices, $W(\mathcal{X})$ can satisfy the following conditions:

$$W(\mathcal{X}_{\text{local}}) \propto 1,$$

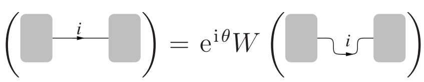


(A21)

where $i,j,\ldots=0,1,\ldots,N_q-1$ label the N_q quasiparticle types and $i=0$ corresponds to the trivial quasiparticle. Note that N_q is also the ground-state degeneracy on the torus. The shaded areas in Eq. (A21) represent other parts of tunneling path. Note that there may be tunneling paths that connect disconnected shaded areas. Here $\mathcal{X}_{\text{local}}$ represents a tunneling path which has a compact support (i.e., there is no path in $\mathcal{X}_{\text{local}}$ that wraps around the torus). In this case $W(\mathcal{X}_{\text{local}})$ represents local perturbations that cannot mix different degenerate ground states on torus.[3] Thus $W(\mathcal{X}_{\text{local}})$ must be proportional to the identity. The second relation in Eq. (A21) implies that the tunneling amplitude $W(\mathcal{X})$ only depends on the topology of the tunneling path. A smooth deformation of tunneling path will not change $W(\mathcal{X})$.

The above conjecture is based on the following consideration. In a class of topologically ordered states—string-net condensed states—the amplitudes of string operators satisfy the above conditions.[6] The quasiparticle tunneling operators behave just like the string operator, and thus we assume they satisfy the same conditions.

Strictly speaking, due to the quasiparticle charge and the external magnetic field, we only have



where θ is a path-dependent phase. However, we can restrict the tunneling paths to be on a properly designed grid, such as a grid formed by squares. We choose the grid such that each square contains k units of flux quanta where k is an integer that satisfies kQ_γ=integer for any γ. In this case, $e^{i\theta}=1$ for the tunneling paths on the grid. Other relations in Eq. (A21) are motivated from tensor category theory.[6–8]

One may notice that rules [Eq. (A21)] are about planar graphs while the tunneling paths are three-dimensional graphs. How can one apply rules of planar graphs to three-dimensional graphs? Here we have picked a fixed direction of projection and projected the three-dimensional tunneling paths to a two-dimensional plane.[42] The rules [Eq. (A21)] apply to such projected planar graphs. Also, the action of tunneling process can only be properly represented by *framed* graphs in three dimensions to take into account the phase factors generated by twisting the quasiparticles. Here we have assumed that there is a way to choose a canonical framing for each tunneling process, such that their projections satisfy Eq. (A21).

The coefficients $(d_i,F^{ijm}_{kln},\omega^k_{ij})$ must satisfy the following self-consistent relations;[6,7]

$$F^{ijk}_{\bar{j}i0}=\frac{v_k}{v_iv_j}n^{\bar{k}}_{ij}$$

$$F^{ijm}_{kln}=F^{lk\bar{m}}_{jin}=F^{jim}_{lk\bar{n}}=F^{imj}_{\bar{k}nl}\frac{v_mv_n}{v_jv_l}$$

$$\sum_{n=0}^{N_q-1}F^{mlq}_{kpn}F^{jip}_{mns}F^{jsn}_{lkr}=F^{jip}_{qkr}F^{riq}_{mls}$$

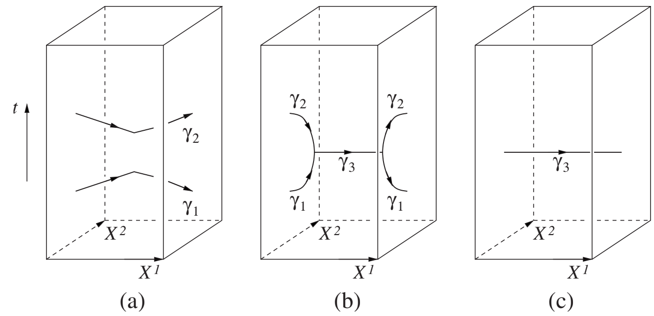


FIG. 10. (a) Two tunneling processes: $A^{(\gamma_1)}$ and $A^{(\gamma_2)}$. (b) The tunneling path of the above two tunneling processes can be deformed according to the fifth relation in Eq. (A21) with $i=\bar{\gamma}_2$, $j=\gamma_2$, $k=\gamma_1$, $k=\bar{\gamma}_1$, $m=0$, and $n=\gamma_3$. (c) The fifth, the fourth, and the third relations in Eq. (A21) can reduce (b) and (c).

$$\omega^m_{js}F^{s\bar{l}i}_{kj\bar{m}}\omega^l_{si}\frac{v_jv_s}{v_m}=\sum_{n=0}^{N}F^{j\bar{i}k}_{\bar{s}n\bar{l}}\omega^n_{sk}F^{j\bar{l}n}_{ks\bar{m}}$$

$$\omega^j_{is}=\sum_{k=0}^{N_q-1}\omega^k_{s\bar{i}}F^{\bar{i}\bar{s}k}_{is\bar{j}^*}. \tag{A22}$$

Here $\bar{i}$ is the antiquasiparticle of i, v_i is defined by $v_i=v_{\bar{i}}=\sqrt{d_i}$, and n^k_{ij} is given by

$$n^k_{ij}=\begin{cases}1 & \text{if } i,j \text{ can fuse into } k\\ 0 & \text{otherwise}.\end{cases}$$

We would like to point out that here we only considered a quasiparticle fusion algebra $\psi_i\psi_j=\sum_k n^k_{ij}\psi_k$ that has a special property $n^k_{ij}=0,1$.

c. Implications of tensor category structure

As the first application of the above algebraic structure, we find that (see Fig. 10)

$$A^{(\gamma_2)}A^{(\gamma_1)}=\sum_{\gamma_3}F^{\bar{\gamma}_2\gamma_2 0}_{\gamma_1\bar{\gamma}_1\gamma_3}d_{\gamma_1}F^{\gamma_3\bar{\gamma}_1\bar{\gamma}_2}_{\gamma_1\bar{\gamma}_3 0}A^{(\gamma_3)}=\sum_{\gamma_3}n^{\gamma_3}_{\gamma_2\gamma_1}A^{(\gamma_3)}. \tag{A23}$$

We see that the algebra of $A^{(\gamma)}$ forms a representation of fusion algebra $\psi_{\gamma_1}\psi_{\gamma_2}=\sum_{\gamma_3}n^{\gamma_3}_{\gamma_1\gamma_2}\psi_{\gamma_3}$. The operators $B^{(\gamma)}=SA^{(\gamma)}S^{-1}$ [see Eq. (A38)] satisfy the same fusion algebra,

$$B^{(\gamma_2)}B^{(\gamma_1)}=\sum_{\gamma_3}n^{\gamma_3}_{\gamma_2\gamma_1}B^{(\gamma_3)}. \tag{A24}$$

As the second application of the above algebraic structure, we can represent the degenerate ground states on torus graphically. One of the degenerate ground state Φ_0 that corresponds to the trivial quasiparticle $\gamma=0$ can be represented by an empty solid torus [see Fig. 11(a)]. We denote such a state as $|0\rangle=\Phi_0$. Other degenerated ground states can be obtained by the action of the $B^{(\gamma)}$ operators,

$$|\gamma\rangle\equiv B^{(\gamma)}|0\rangle. \tag{A25}$$

From Eq. (A20), we see that Φ_γ and $|\gamma\rangle$ are related,

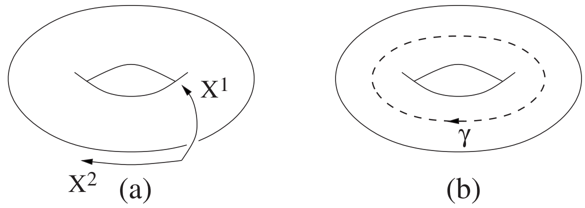


FIG. 11. (a) The ground state $\Phi_{\gamma=0}$ on a torus that corresponds to the trivial quasiparticle can be represented by an empty solid torus. (b) The other ground state Φ_γ that corresponds to a type γ quasiparticle can be represented by an solid torus with a loop of type γ in the center.

$$b_\gamma \Phi_\gamma = |\gamma\rangle.$$

Since $|\gamma\rangle$ is created by the tunneling operator $B^{(\gamma)}$, $|\gamma\rangle$ can be represented by adding a loop that corresponds to the $B^{(\gamma)}$ operator to the center of the solid torus [see Fig. 11(b)].

$|\gamma\rangle$ is a natural basis for tensor category theory. The matrix elements of $A^{(\gamma)}$ and $B^{(\gamma)}$ have simple forms in such a basis. From Eq. (A24), we see that

$$B^{(\gamma_2)}B^{(\gamma_1)}|0\rangle = B^{(\gamma_2)}|\gamma_1\rangle = \sum_{\gamma_3} n^{\gamma_3}_{\gamma_2\gamma_1}|\gamma_3\rangle.$$

Therefore, in the basis $|\gamma\rangle$, the matrix elements of $B^{(\gamma_2)}$ are given by the coefficients of fusion algebra,

$$(B^{(\gamma_2)})_{\gamma_3\gamma_1} = n^{\gamma_3}_{\gamma_2\gamma_1}. \tag{A26}$$

The action of $A^{(\gamma')}$ on $|\gamma\rangle$ is represented by Fig. 12. From Fig. 13, we find that

$$A^{(\gamma')}|\gamma\rangle = \frac{S^{\mathrm{TC}}_{\bar{\gamma}\gamma'}}{d_\gamma}|\gamma\rangle, \tag{A27}$$

where $S^{\mathrm{TC}}_{\gamma_1\gamma_2}$ is the amplitude of two linked local loops (see Fig. 14). $S^{\mathrm{TC}}_{\gamma_1\gamma_2}$ satisfies

$$S^{\mathrm{TC}}_{\gamma_1\gamma_2} = S^{\mathrm{TC}}_{\gamma_2\gamma_1}. \tag{A28}$$

We see that $A^{(\gamma')}$ is diagonal in the $|\gamma\rangle$ basis. Let $a^{(\gamma')}_\gamma$ be the eigenvalues of $A^{(\gamma')}$. We see that

$$a^{(\gamma')}_\gamma = \frac{S^{\mathrm{TC}}_{\bar{\gamma}\gamma'}}{d_\gamma} = \frac{S^{\mathrm{TC}}_{\gamma'\bar{\gamma}}}{S^{\mathrm{TC}}_{0\bar{\gamma}}}. \tag{A29}$$

Using the tensor category theory (A21), one can show that[7]

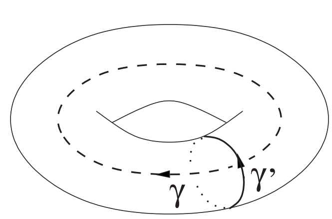


FIG. 12. The graphic representation of $A^{(\gamma')}|\gamma\rangle$.

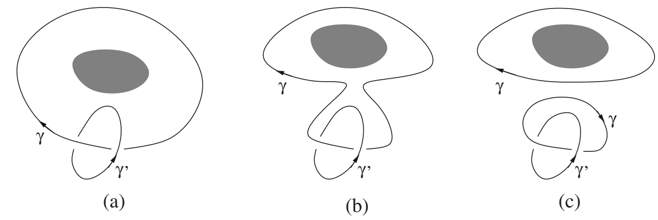


FIG. 13. (a) The graphic representation of $A^{(\gamma')}|\gamma\rangle$. (b) The graphic representation of $A^{(\gamma')}|\gamma\rangle$. (c) The fifth and the fourth relations in Eq. (A21) can deformed the graph in (b) to the graph in (c). The shaded area represents the hole of the torus.

$$\frac{S^{\mathrm{TC}}_{\gamma_1\gamma}S^{\mathrm{TC}}_{\gamma_2\gamma}}{S^{\mathrm{TC}}_{0\gamma}} = \sum_{\gamma_3} n^{\gamma_3}_{\gamma_1\gamma_2}S^{\mathrm{TC}}_{\gamma_3\gamma} = \sum_{\gamma_3} S^{\mathrm{TC}}_{\gamma\gamma_3}B^{(\gamma_1)}_{\gamma_3\gamma_2}, \tag{A30}$$

which can be rewritten as

$$\frac{S^{\mathrm{TC}}_{\gamma_1\gamma}}{S^{\mathrm{TC}}_{0\gamma}}\delta_{\gamma\gamma'} = \sum_{\gamma_3\gamma_2} S^{\mathrm{TC}}_{\gamma\gamma_3}B^{(\gamma_1)}_{\gamma_3\gamma_2}[(S^{\mathrm{TC}})^{-1}]_{\gamma_2\gamma'} = A^{(\gamma_1)}_{\bar{\gamma}\bar{\gamma}'}. \tag{A31}$$

In the operator form, the above becomes

$$CA^{(\gamma_1)}C = S^{\mathrm{TC}}B^{(\gamma_1)}(S^{\mathrm{TC}})^{-1}, \tag{A32}$$

where C is the charge-conjugation operator $C|\gamma\rangle = |\bar{\gamma}\rangle$. We see that S^{TC} can change the tunneling operator $B^{(\gamma_1)}$ to $A^{(\gamma_1)}$. Equation (A30) can also be rewritten as (assume S^{TC} is invertible)

$$\sum_\gamma \frac{S^{\mathrm{TC}}_{\gamma\gamma_1}S^{\mathrm{TC}}_{\gamma\gamma_2}[(S^{\mathrm{TC}})^{-1}]_{\gamma_3\gamma}}{S^{\mathrm{TC}}_{\gamma 0}} = n^{\gamma_3}_{\gamma_1\gamma_2}. \tag{A33}$$

We note that the above expression is invariant under $S^{\mathrm{TC}}_{\gamma\gamma'} \to f_\gamma S^{\mathrm{TC}}_{\gamma\gamma'}$. We also note that the above expression implies that

$$n^{\gamma_3}_{\gamma_1\gamma_2} = n^{\gamma_3}_{\gamma_2\gamma_1}.$$

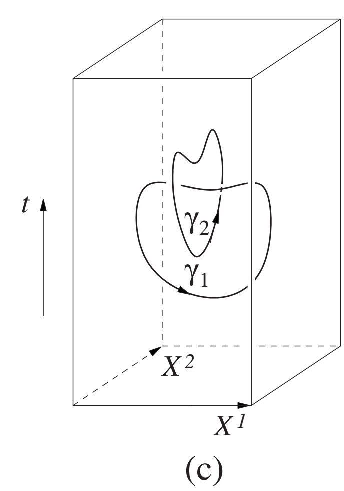


FIG. 14. The amplitude of two linked local loops is a complex number $S^{\mathrm{TC}}_{\gamma_1\gamma_2}$.

T_1 and T_2 generate the translation symmetry of the torus. We expect that T_1 and T_2 commute with the algebraic structure of the tensor category. Thus we expect that Eq. (47) keeps the same form in the new basis $|\gamma\rangle$,

$$T_1|\gamma\rangle = e^{i2\pi Q_\gamma}|\gamma\rangle, \quad T_2|\gamma\rangle = |\gamma'\rangle, \tag{A34}$$

where γ' is the quasiparticle described by the canonical occupation distribution $n_{\gamma';l}=n_{\gamma;(l-1)\%m}$.

d. Quasiparticle tunneling operators under modular transformation

The transformations $W^{(\gamma)}_{(n_1,n_2)}$ induced by quasiparticle tunneling processes have certain algebraic relation with the modular transformations $U(M)$. From Fig. 7, we see that the modular transformation M_T changes $W^{(\gamma)}_{(n_1,n_2)}$ to $W^{(\gamma)}_{(n_1+n_2,n_2)}$,

$$TW^{(\gamma_1)}_{(n_1,n_2)} = W^{(\gamma_1)}_{(n_1+n_2,n_2)}T. \tag{A35}$$

Since the modular transformation M_S generates a 90° rotation, we find

$$SW^{(\gamma_1)}_{(n_1,n_2)} = W^{(\gamma_1)}_{(-n_2,n_1)}S. \tag{A36}$$

Here M_T and M_S are given by Eq. (49) and T and S are given by Eq. (50). Also since the modular transformation $M=-1$ generates a 180° rotation, we find

$$CW^{(\gamma_1)}_{(n_1,n_2)} = W^{(\gamma_1)}_{(-n_2,-n_1)}C. \tag{A37}$$

In terms of $A^{(\gamma_1)}$ and $B^{(\gamma_1)}$ we can rewrite Eqs. (A35)–(A37) as

$$TA^{(\gamma_1)} = A^{(\gamma_1)}T,$$

$$TB^{(\gamma_1)} = W^{(\gamma_1)}_{(1,1)}T,$$

$$SA^{(\gamma_1)} = B^{(\gamma_1)}S,$$

$$SB^{(\gamma_1)}C = CA^{(\gamma_1)}S. \tag{A38}$$

Let us assume that the set of the quasiparticle operators $A^{(\gamma)}$ can resolve all the degenerate ground states $|\gamma\rangle$, i.e., no two degenerate ground states share the common set of eigenvalues for the operators $A^{(\gamma)}$. In this case, the commutation relation $TA^{(\gamma_1)}=A^{(\gamma_1)}T$ implies that T is diagonal in the $|\gamma\rangle$ basis. We will fix the overall phase factor of T by choosing $T_{00}=1$.

The operator $C=S^2$ is a charge-conjugation operator. Its action on $|\gamma\rangle$ is given by [see Eq. (A18)]

$$C|\gamma\rangle = |\bar{\gamma}\rangle.$$

Compare $S^{-1}CA^{(\gamma_1)}CS=B^{(\gamma_1)}$ with Eq. (A32), we find that $S^{\mathrm{TC}}=FS$, where F is a diagonal matrix $F_{\gamma\gamma'}=f_\gamma\delta_{\gamma\gamma'}$ in the $|\gamma\rangle$ basis. Using $S^{\mathrm{TC}}=FS$, we can rewrite Eqs. (A29) and (A33) as

$$\frac{f_\gamma S_{\gamma\gamma_1}}{f_\gamma S_{\gamma 0}} = \frac{S_{\gamma\gamma_1}}{S_{\gamma 0}} = a^{(\gamma_1)}_{\bar{\gamma}}, \tag{A39}$$

$$\sum_\gamma \frac{S_{\gamma\gamma_1}S_{\gamma\gamma_2}(S^{-1})_{\gamma_3\gamma}}{S_{\gamma 0}} = n^{\gamma_3}_{\gamma_1\gamma_2}. \tag{A40}$$

We see that $A^{(\gamma)}$ and $B^{(\gamma)}$ can be determined from S.

Since S^{TC} is symmetric, $S^{\mathrm{TC}}_{0\gamma}=d_\gamma>0$ and $S^{\mathrm{TC}}_{00}=1$, once we know S, we can use those conditions to fix F. Thus we can determine S^{TC} from S. Once we know S^{TC}, we can also calculate the CFT scaling dimension h_γ for the quasiparticle γ (see Appendix, Sec. 7) up to an integer,[7]

$$S^{\mathrm{TC}}_{\gamma_1\gamma_2} = \sum_{\gamma_3} n^{\gamma_3}_{\gamma_1\gamma_2} e^{i2\pi(h_{\gamma_3}-h_{\gamma_1}-h_{\gamma_2})} d_{\gamma_3}. \tag{A41}$$

7. Relation to conformal field theory

The symmetric polynomial Φ can be written as a correlation function of vertex operators $V_e(z)$ in a conformal field theory (CFT),[39,43,44]

$$\Phi(\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h_N}\langle V(z_\infty)\prod_i V_e(z_i)\rangle. \tag{A42}$$

V_e (which will be called an electron operator) has a form

$$V_e(z) = \psi(z)e^{i\phi(z)/\sqrt{\nu}},$$

where ψ is a simple-current operator and $e^{i\phi(z)/\sqrt{\nu}}$ is the vertex operator on a Gaussian model with a scaling dimension $h=\frac{1}{2\nu}$. The scaling dimension h^{sc}_a of $\psi_a(z)\equiv[\psi(z)]^a$ has being calculated from the pattern of zeros $\{S_a\}$ in Ref. 12,

$$h^{\mathrm{sc}}_a = S_a - \frac{aS_n}{n} + \frac{am}{2} - \frac{a^2m}{2n}. \tag{A43}$$

The quasiparticle state Φ_γ can also be expressed as a correlation function in a CFT,

$$\Phi_\gamma(\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h^q_N}\langle V_q(z_\infty)V_\gamma(0)\prod_i V_e(z_i)\rangle. \tag{A44}$$

(Note that Φ_γ has a quasiparticle at $z=0$.) Here V_γ is a quasiparticle operator in CFT which has a form

$$V_\gamma(z) = \sigma_\gamma(z)e^{i\phi(z)Q_\gamma/\sqrt{\nu}}, \tag{A45}$$

where $\sigma_\gamma(z)$ is a "disorder" operator in the CFT generated by the simple-current operator ψ. Different quasiparticles labeled by different γ will correspond to different disorder operators.

Let us introduce a quantitative way to characterize the quasiparticle operator. We first fuse the quasiparticle operator with a electron operators,

$$V_{\gamma;a}(z) = V_\gamma V_a = \sigma_{\gamma;a}(z)e^{i\phi(z)Q_{\gamma;a}/\sqrt{\nu}}$$

$$\sigma_{\gamma;a} = \sigma_\gamma\psi_a, \quad Q_{\gamma;a} = Q_\gamma + a, \tag{A46}$$

where $V_a\equiv(V_e)^a=\psi_a e^{ia\phi(z)a/\sqrt{\nu}}$. Then, we consider the operator product expansion (OPE) of $V_{\gamma;a}$ with V_e,

$$V_e(z)V_{\gamma;a}(w) = (z-w)^{l^{\mathrm{CFT}}_{\gamma;a+1}}V_{\gamma;a+1}(w). \tag{A47}$$

Let h_a, h_γ, and $h_{\gamma;a}$ be the scaling dimensions of V_a, V_γ, and $V_{\gamma;a}$, respectively. We have

$$l^{\text{CFT}}_{\gamma;a+1} = h_{\gamma;a+1} - h_1 - h_{\gamma;a}. \tag{A48}$$

Since the quasiparticle wave function $\Phi_\gamma(\{z_i\})$ must be a single valued function of z_i's, this requires that $l^{\text{CFT}}_{\gamma;a}$ must be integers. The sequence of integers $\{l^{\text{CFT}}_{\gamma;a}\}$ gives us a quantitative way to characterize quasiparticle operators V_γ in CFT.

From the occupation distribution description of the quasiparticle γ introduced in Sec. III B, we see that a quasiparticle can also be characterized by another sequence of integers $\{l_{\gamma;a}\}$. What is the relation between the two sequences of integers, $\{l^{\text{CFT}}_{\gamma;a}\}$ and $\{l_{\gamma;a}\}$, which characterize the same set of quasiparticles? From Eq. (A47), we see that $l^{\text{CFT}}_{\gamma;a}$ is the order of zeros as we move an electron z_i toward a quasiparticle γ fused with a electrons. Thus $l^{\text{CFT}}_{\gamma;a}$ is the order of zero $D_{\gamma;a,1}$ introduced in Sec. III A. From Eq. (18) and $S_1=0$, we find that $l^{\text{CFT}}_{\gamma;a}$ in the above OPE is given by $l^{\text{CFT}}_{\gamma;a}=S_{\gamma,a}-S_{\gamma,a-1}$. Thus the two sequences, $\{l^{\text{CFT}}_{\gamma;a}\}$ and $\{l_{\gamma;a}\}$, are identical, $\{l^{\text{CFT}}_{\gamma;a}\}=\{l_{\gamma;a}\}$. In the rest of the paper, we will drop the superscript CFT in $l^{\text{CFT}}_{\gamma;a}$.

Now let us calculate the quasiparticle charge Q_γ [see Eq. (A45)] from the sequence $\{l_{\gamma;a}\}$ within the CFT. Using $l_{\gamma,a}=S_{\gamma,a}-S_{\gamma,a-1}$, we can rewrite Eq. (A48) as $h_{\gamma;a+1}-h_{\gamma,a}=S_{\gamma;a+1}-S_{\gamma;a}+h_1$. Thus $h_{\gamma;a}-h_\gamma=S_{\gamma;a}+ah_1$, where we have used $S_{\gamma;0}=S_\gamma=0$. Using $h_1=h^{\text{sc}}_1+\frac{1}{2\nu}=\frac{m}{2}-\frac{S_n}{n}$, we find

$$h_{\gamma;a} - h_\gamma = S_{\gamma;a} + a\left(\frac{m}{2} - \frac{S_n}{n}\right).$$

Since $\sigma_{\gamma,n}=\sigma_\gamma$, we have

$$h_{\gamma;n} - h_\gamma = \frac{(Q_\gamma+n)^2 - Q_\gamma^2}{2\nu} = S_{\gamma;n} + n\left(\frac{m}{2} - \frac{S_n}{n}\right).$$

Thus

$$Q_\gamma = \frac{S_{\gamma;n} - S_n}{m} = \frac{1}{m}\sum_{a=1}^{n}(l_{\gamma;a} - l_a),$$

which agrees with Eq. (27).

Let $h^{\text{sc}}_{\gamma;a}$ be the scaling dimension of $\sigma_\gamma\psi_a$. We see that $h^{\text{sc}}_{\gamma;a}=h_{\gamma;a}-\frac{(Q_\gamma+a)^2}{2\nu}$ and

$$h^{\text{sc}}_{\gamma;a} - h^{\text{sc}}_\gamma = S_{\gamma;a} - \frac{m}{2n}a^2 + \left(\frac{m}{2} - \frac{S_{\gamma;n}}{n}\right)a.$$

If

$$s_{\gamma;a} = S_{\gamma;a} - \frac{m}{2n}a^2 + \left(\frac{m}{2} - \frac{S_{\gamma;n}}{n}\right)a = s_{\gamma;a+n},$$

we can rewrite the above as

$$h^{\text{sc}}_{\gamma;a} = h^{\text{sc}}_\gamma + s_{\gamma;a}. \tag{A49}$$

We see that the simple-current part of CFT is determined by $s_{\gamma;a}$, $a=1,\ldots,n-1$, only. In particular, if $\sigma_{\gamma'}$ and σ_γ are related by a simple-current operator, $\sigma_{\gamma'}=\sigma_\gamma\psi_a$, then the scaling dimension of $\sigma_{\gamma'}$ can be calculated from that of σ_γ: $h^{\text{sc}}_{\gamma'}=h^{\text{sc}}_\gamma+s_{\gamma;a}$.

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